Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 22, 2003*
Accepted: *October 13, 2003*

Predictions for PP-wave string amplitudes from perturbative SYM

Umut Gürsoy

Center for Theoretical Physics
Laboratory for Nuclear Science and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.
E-mail: umut@mit.edu

Abstract: The role of general two-impurity multi-trace operators in the BMN correspondence is explored. Surprisingly, the anomalous dimensions of all two-impurity multi-trace BMN operators to order $g_2^2\lambda'$ are completely determined in terms of single-trace anomalous dimensions. This is due to suppression of connected field theory diagrams in the BMN limit and this fact has important implications for some string theory processes on the PP-wave background. We also make gauge theory predictions for the matrix elements of the light-cone string field theory hamiltonian in the two string-two string and one string-three string sectors.

Keywords: Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

PP-wave/SYM duality [1] relates string states in IIB string theory on the PP-wave background to a particular class of observables in $\mathcal{N}=4$ SYM. These observables, called BMN operators, are singled out in the Hilbert space of SYM by a "modified" 't Hooft limit,

$$N \to \infty\,, \qquad \text{with } \frac{J}{\sqrt{N}} \text{ and } g_{\rm YM} \text{ fixed.} \tag{1.1}$$

In detail, a single-trace BMN operator which involves two scalar impurities is,

$$O_n^J = \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} \mathrm{Tr}(\phi Z^l \psi Z^{J-l}) e^{\frac{2\pi i n l}{J}}\,. \tag{1.2}$$

In the *free* string theory it is dual to a single-string state with two-excitations,

$$\alpha_n^1 \alpha_{-n}^2 |p^+, 0\rangle$$

where the light-cone momentum and R-charge J are related by,

$$\mu p^+ \alpha' = \frac{J}{\sqrt{g_{\rm YM}^2 N}} \equiv (\lambda')^{-1/2}\,.$$

Similarly an i-trace BMN operator is formed as,

$$O_i^J =: O_n^{J_1} O^{J_2} \cdots O^{J_i} : , \tag{1.3}$$

from (1.2) and the BPS operators,

$$O^J = \frac{1}{\sqrt{JN^J}} \mathrm{Tr}(Z^J) \, . \tag{1.4}$$

Here $J = J_1 + \cdots + J_i$. In the *free* string theory, this operator naturally maps onto an i-string state of the form,

$$\psi_i \equiv \alpha_n^1 \alpha_{-n}^2 |p_1^+, 0\rangle \otimes |p_2^+, 0\rangle \otimes \cdots |p_i^+, 0\rangle \, .$$

A striking aspect of PP-wave/SYM correspondence is that, in a regime where both the effective GT coupling λ' and string coupling g_s are small, one has *a duality between effectively weakly coupled gauge theory and perturbative string theory.* This goes beyond the aforementioned duality between observables in SYM and string states on pp-wave background in the free string theory and provides an explicit map between gauge and string interactions. However, a clear understanding of the correspondence at the level of interactions still remains as an interesting challenge. One essential reason which hinders a complete understanding is the fact that, while states with different number of strings are orthogonal in SFT Hilbert space at all orders in g_s, one gets a non-trivial mixing between BMN operators of different number of traces when one turns on the genus-counting parameter of SYM,

$$g_2 = \frac{J^2}{N} \, ,$$

which correspond to g_s on the string side. This fact can be summarized by the following formula,

$$\langle \bar{O}_i^J O_j^J \rangle_{g_2} \sim g_2^{|i-j|} \, , \qquad i \neq j \, .$$

This is true even in the free theory, i.e. for $\lambda' = 0$.

Recently, several important steps were taken in relation to this problem. A natural route to take is to identify the *dynamical* generators P^- and $\Delta - J$ as,

$$\frac{2P^-}{\mu} = \Delta - J$$

also for non-zero values of g_2 and g_s [6]. Since these operators act on completely different Hilbert spaces, an unambiguous identification is achieved only by equating the eigenvalues of P^- and $\Delta - J$ in the corresponding sectors of the Hilbert spaces. In case of one-string states this problem was considered in a number of papers. On the gauge theory (GT) side, $\mathcal{O}(g_2^2)$ eigenvalue of BMN operators that correspond to single-string states was obtained in [6, 7, 8]. On the string theory side, one-string eigenvalue of P^- at $\mathcal{O}(g_s^2)$ was first addressed in [9] where a computation that partially uses the language of String Bit Formalism (SBF) [10] was performed and exact agreement with the GT result was reported. As noted in that paper however, an ultimate check of the correspondence requires a purely string field theory (SFT) computation [11]. Very recently, this calculation was carried out in [12] and also perfect agreement with GT eigenvalue was established.[1]

[1]Up to an ambiguity which arise from a particular truncation of the intermediate string-states.

JHEP10(2003)027

Apart from the correspondence of eigenvalues, it is quite desirable to have an identification of the *matrix elements* of P^- and $\Delta - J$. This, of course requires, first to establish an isomorphism between the complete bases that these elements are evaluated in. As the BMN operators of different types mix with each other even in the free SYM, it becomes essential, first to give a characterization of a particular "string basis" in GT in which BMN operators $\tilde{O}_i$, at the free level form an orthonormal basis.

One can fix the ambiguity in identification of the string basis e.g. by requiring the matrix elements of $\Delta - J$ at $\mathcal{O}(g_2)$ between single and double-trace modified BMN operators match with $\langle\psi_1|P^-|\psi_2\rangle$ where ψ_i denote an i-string state. This question was addressed very recently by two seemingly different methods, [9] and [13] (whose compatibility we demonstrate at $\mathcal{O}(g_2^2)$ in appendix B.) After fixing the string basis at a certain order in g_2, evaluation of various matrix elements of $\Delta - J$ at the next order provides predictions for SFT computations. For example after fixing the basis transformation at $\mathcal{O}(g_2^2)$ as explained above one should be able to match the matrix elements $\langle\tilde{O}_i|\Delta - J|\tilde{O}_{i+2}\rangle$ with $\langle\psi_i|P^-|\psi_{i+2}\rangle$ whose leading order contributions are at $\mathcal{O}(g_2^2)$.

So far these problems were considered only in case of single and double-trace BMN operators. In this note, we will take the first step in addressing the role played by higher order multi-trace operators in the perturbative PP-wave/SYM correspondence. We work out the case of triple-trace BMN operators in detail. But we will be able to draw some general conclusions which hold for *all* two-impurity multi-trace BMN operators.

Let us first address a subtlety in the determination $\Delta - J$ eigenvalue in GT which was discussed in [8] and also relevant in the analysis of [7]. The mixing between BMN operators of different number of traces indicates that O_i does not have the required transformation properties under dilatation for non-zero g_2. Therefore the first step in determination of the eigenvalue of $\Delta - J$ is to construct an orthogonal set of BMN operators, $\tilde{O}_i$ which takes the effects of operator mixing into account. Then one can obtain the eigenvalue Δ_i essentially by using the information contained in the free and $\mathcal{O}(\lambda')$ parts of the two-point function, $\langle\bar{\tilde{O}}_i^J \tilde{O}_j^J\rangle$. In [8] it was pointed out that this diagonalization procedure is equivalent to first-order non-degenerate perturbation theory where one first obtains the eigen-operators of $\Delta - J$ at $\mathcal{O}(g_2)$, then works out the $\mathcal{O}(g_2^2)$ eigenvalue as the second step in perturbation theory. However to assure the validity of this procedure, one needs to establish the non-degeneracy at $\mathcal{O}(g_2^2)$. In [7, 8] it was found that the non-degeneracy holds at $\mathcal{O}(g_2)$ thanks to very delicate cancellations and the following result for the anomalous dimension was obtained,

$$\Delta_1 = \frac{g_2^2}{4\pi^2}\left(\frac{1}{12} + \frac{35}{32\pi^2 n^2}\right). \tag{1.5}$$

It was also pointed out that if perturbation theory becomes degenerate at the next order in g_2, the result for the eigenvalue would only be valid in the very special case of world-sheet momenta $n = 1$. In section 2, we perform the required analysis in second-order perturbation theory by taking into account the mixing with the triple-trace operators and show that non-degenerate perturbation theory actually becomes invalid at $\mathcal{O}(g_2^2)$. This phenomenon was first observed in [19]. This result casts some doubt on the validity of eq. (1.5) for $n > 1$ and necessiates the use of degenerate perturbation theory at $\mathcal{O}(g_2^2)$.[2]

However, as we briefly discuss at the end of section 2, one can show that the use of degenerate in place of non-degenerate perturbation theory do not alter the previously obtained results drastically: for finite J, the degenerate subspace that contain the single-trace operators consists of single-trace, triple-trace, 5-trace, ..., $J =$ trace operators (J is chosen as an odd number for convenience). One can argue that there always exist an eigenstate in this subspace which, in the limit, $J \to \infty$ continuously transformed into $\tilde{O}_n^J$, and its eigenvalue tends to eq. (1.5) in this limit. Futhermore there exist eigenstates in the seperate degenerate subspaces of operators with odd and even numbered traces whose eigenvalues in the BMN limit become the anomalous dimensions associated with $\tilde{O}_i^J$. It is possible to demostrate this fact in a variety of effective models. We would like to give details of this interesting phenomenon along with new results associated with the use of degenerate perturbation theory and the leave the general proof of the aforementioned fact in a future work. Therefore our assertion can be taken as a conjecture in this paper.

After establishing the validity of our method for the aforementioned particular eigenstates, one can ask for the $\mathcal{O}(g_2^2)$ eigenvalues of higher trace BMN operators. We consider this problem in section 4. A rigorous investigation yields an unexpected result: *Eigenvalues of all multi-trace BMN operators are solely determined by the eigenvalue* Δ_1 *of single-trace operator as,*

$$\Delta_i = \left(\frac{J_1}{J}\right)^2 \Delta_1 \,.$$

This result is essentially due to the suppression of connected GT correlators $\langle \bar{\tilde{O}}_i \tilde{O}_j \rangle$ by a power of J as $J \to \infty$ in the BMN limit. It is found that disconnected GT diagrams are less suppressed in this limit and in fact only non-zero contributions to a generic correlator of BMN operators arise from fully disconnected pieces. Note however that, the connected correlators will contribute to eigenvalues to higher order in g_2.

Utilizing the correspondence of $\Delta - J$ with P^- in the string theory we show that this fact translates into the absence of $\mathcal{O}(g_s^2)$ contact terms between states higher than single-string states. If the correspondence with P^- at the level of matrix elements holds, this also implies that a particular class of tree-level string processes that would contribute to the matrix elements on the PP-wave are suppressed in the large μ limit. Since we consider BMN operators with only two scalar impurities, our conclusions are valid only for the processes in which the external string states have excitations along $i = 1, 2, 3, 4$ transverse directions (that correspond to scalar impurities [1])

It is also interesting to investigate the the duality of P^- and $\Delta - J$ at the level of matrix elements. Using the method of [13] to fix the basis transformation into "string basis" at $\mathcal{O}(g_2^2)$, we study the correlators of double and triple operators in this basis and obtain predictions for the matrix elements of P^- at $\mathcal{O}(g_s^2)$, in double-double and single-triple sectors. These matrix elements are given by remarkably simple expressions and

[2]In the first version of this paper we made the opposite conclusion due to an unfortunate error in section 2. We thank C. Kristjansen for pointing this out.

solely determined by the "non-contractible" contribution to $\langle \bar{O}_i O_j \rangle$ correlator just as in the case of single-single matrix element [9, 13]. On the ST side, in the single-string sector the matrix element is determined by the "contact" interaction between two single-string states [12, 9]. Our study suggests a generalization of this fact: a one-to-one map between non-contractible contributions to GT correlation functions and contact interactions of the corresponding states in SFT. These are explicit gauge theory results that are subject to check by a direct SFT calculation.

We organize the paper as follows. In the next section we demonstrate the invalidity of non-degenerate perturbation theory in determining the eigen-operators and eigenvalues of $\Delta - J$. Taking into account the mixing with triple-trace operators we obtain the mostly single-trace eigen-operator at $\mathcal{O}(g_2^2)$. We briefly outline our conjecture that use of degenerate and non-degenerate perturbation theory leads us to the same results concerning the anomalous dimensions of particular eigen-states that correspond to $\tilde{O}_n^J$ and $\tilde{O}_i^J$ in the BMN limit. This section also introduces necessary notation and presents single-double, single-triple and double-triple trace BMN correlators. In section 3, we discuss the scaling behavior of arbitrary multi-trace correlators of BMN operators with g_2 and J. We demonstrate that connected contributions to all of the correlators of this sort are suppressed as $J \to \infty$. In section 4, we utilize this result to obtain the anomalous dimension of an i-trace BMN operator at $\mathcal{O}(g_2^2\lambda')$. We also discuss some implications for the corresponding processes in string theory in this section. Last section studies the duality between P^- and $\Delta - J$ at the level of matrix elements.

Appendix A proves the scaling behaviour that we discuss in section 3. Appendix B deals with the basis transformation which takes from the BMN basis into string basis in GT. Using the inputs from [9] and [13] we derive new decomposition identities relating various multi-trace inner products with the product of smaller order inner products. In particular, the free single-triple inner product decomposes into single-double and double-triple inner products as,

$$G^{13} = \frac{1}{2} G^{12} G^{23} \,.$$

Similarly we derive the identity,

$$G^{22} = \frac{1}{2}(G^{21}G^{12} + G^{23}G^{32})$$

and discuss immediate generalizations. We emphasize that these identities are derived by relating the basis transformations proposed in [13] and [9], therefore subject to explicit GT computations. These computations which involve non-trivial summations are outlined in appendix C and these identities are proven there. In this appendix, we also explain evaluation of other sums that are used in sections 2, 4 and 5. Appendix D computes $\mathcal{O}(g_2^2)$ and $\mathcal{O}(g_2^2\lambda')$ contributions to single-triple and $\mathcal{O}(g_2)$ and $\mathcal{O}(g_2\lambda')$ contributions to double-triple correlation functions.

2. Operator mixing at g_2^2 level

In this section we shall carry out the diagonalization procedure of the multi-trace BMN operators including the mixing with triple trace operators. This is achieved by extending

JHEP10(2003)027

the method of [8] to include the $\mathcal{O}(g_2^2)$ and $\mathcal{O}(g_2^2\lambda')$ effects in the diagonalization. In [8], it was shown that the procedure of determining the eigenvectors and eigenvalues of the mixing matrix of single and double trace operators (which is $\mathcal{O}(g_2\lambda')$) is equivalent to first order non-degenerate perturbation theory. To include the mixing with triple trace operators one needs to go one step further in the perturbation expansion, i.e. to second order perturbation theory.

Let us first outline the method of [8] briefly. Consider the eigenvalue problem,

$$M^i_j e^j_{(k)} = \lambda_{(k)} e^i_{(k)} \tag{2.1}$$

where M is the $3 \times \infty$ dimensional mixing matrix of single, double and triple trace operators.[3] Here i, j is a collective index labeling the state of a BMN operator, e.g. for a triple trace, $i = \{m, y, z\}$ where $y = J_1/J$ and $z = J_2/J$ in (1.3) for $i = 3$. The order in g_2 of the various blocks of M is indicated by,

$$M = \begin{pmatrix} 1 & g_2 & g_2^2 \\ g_2 & 1 & g_2 \\ g_2^2 & g_2 & 1 \end{pmatrix} .$$

Therefore it is possible to solve the eigenvalue problem order by order in g_2. Expanding M, e and λ as

$$\begin{aligned} M^i_j &= \rho_i \delta^i_j + g_2 {M^{(1)}}^i_j + g_2^2 {M^{(2)}}^i_j \\ e^i_{(k)} &= \delta^i_k + g_2 {e^{(1)}}^i_{(k)} + g_2^2 {e^{(2)}}^i_{(k)} \\ \lambda_{(k)} &= \lambda^{(0)}_{(k)} + g_2 \lambda^{(1)}_{(k)} + g_2^2 \lambda^{(2)}_{(k)}, \end{aligned}$$

we obtain,

$$\begin{aligned} 0 = &\left(\rho_k - \lambda^{(0)}_{(k)}\right) \delta^i_k + g_2 \left(\rho_i {e^{(1)}}^i_{(k)} + {M^{(1)}}^i_k - \lambda^{(0)}_{(k)} {e^{(1)}}^i_{(k)} - \delta^i_k \lambda^{(1)}_{(k)}\right) + \\ &+ g_2^2 \left(\rho_i {e^{(2)}}^i_{(k)} + {M^{(2)}}^i_k + {M^{(1)}}^i_j {e^{(1)}}^j_{(k)} - \lambda^{(0)}_{(k)} {e^{(2)}}^i_{(k)} - \lambda^{(1)}_{(k)} {e^{(1)}}^i_{(k)} - \lambda^{(2)}_{(k)} \delta^i_k\right) . \end{aligned} \tag{2.2}$$

At zeroth order one gets $\lambda^{(0)}_{(k)} = \rho_k$. Using this in the next order for $i \neq k$ yields the first order eigenvectors,

$${e^{(1)}}^i_{(k)} = \frac{{M^{(1)}}^i_k}{\rho_k - \rho_i} ,$$

whereas for $i = k$ we learn that $\lambda^{(1)}_{(k)} = 0$.

Using these results, $\mathcal{O}(g_2^2)$ piece of (2.2) for $i = k$ gives,

$$\lambda^{(2)}_{(k)} = \sum_j \frac{{M^{(1)}}^k_j {M^{(1)}}^j_k}{\rho_k - \rho_j} , \tag{2.3}$$

[3] In the next section we explain why BPS type double and triple trace operators do not affect the following discussion.

and for $i \neq k$ we obtain the second order contribution to the eigenvectors,

$$e^{(2)}{}^{i}_{(k)} = \frac{1}{\rho_k - \rho_i}\left(M^{(2)}{}^{i}_{k} + \sum_j \frac{M^{(1)}{}^{i}_{j} M^{(1)}{}^{j}_{k}}{\rho_k - \rho_j}\right). \tag{2.4}$$

Using above expressions for $e^{(1)}$ and $e^{(2)}$, we obtain the single-trace eigen-operator modified at $\mathcal{O}(g_2^2)$ as,

$$\begin{aligned}\tilde{O}^J_n &= O^J_n + g_2 \sum_{my} \frac{\Gamma^{my}_n}{n^2 - (m/y)^2} O^J_{my} + \\ &+ g_2^2 \sum_{myz} \frac{1}{n^2 - (m/y)^2}\left(\Gamma^{myz}_n + \sum_{m'y'} \frac{\Gamma^{myz}_{m'y'}\Gamma^{m'y'}_n}{n^2 - (m'/y')^2}\right) O^J_{myz}\,. \end{aligned}\tag{2.5}$$

Here,

$$\Gamma^i_j = G^{ik}\Gamma_{kj}$$

are the matrix of anomalous dimensions where G^{ij} denotes the inverse metric on the field space. The metric, G_{ij} is determined by the correlation functions $\langle \bar{O}_i O_j \rangle$ at the free level whereas $\mathcal{O}(\lambda')$ radiative corrections to this correlator yield Γ_{ij}. In short,

$$\langle \bar{O}_i O_j \rangle = G_{ij} - \lambda' \Gamma_{ij} \ln(x^2 \Lambda^2)\,. \tag{2.6}$$

G and Γ should be expanded in powers of g_2. Instead of denoting the order in g_2 on G and Γ, we will show g_2 dependence explicitly in what follows.

Again, using above expressions for first and second order eigenvectors, $e^{(1)}$ and $e^{(2)}$, one obtains the double-trace eigen-operator as,

$$\tilde{O}^J_{ny} = O^J_{ny} + g_2 \sum_m \frac{\Gamma^m_{ny}}{(n/y)^2 - m^2} O^J_m + g_2 \sum_{my'z'} \frac{\Gamma^{my'z'}_{ny}}{(n/y)^2 - (m/y')^2} O^J_{my'z'}\,. \tag{2.7}$$

A very important point is to notice that these expressions are valid when the coefficients in front of O_i on the r.h.s. are finite for all values of external and internal momenta. In particular one needs to check the finiteness of (2.5) when the incoming and outgoing world-sheet energies are equal, $n = \pm(m/y)$ and also at $n = \pm(m'/y')$ for the internal denominator in the third term. Note that, the danger of degeneracy is absent only for the case $n = 1$. Therefore without checking the finiteness at $\mathcal{O}(g_2^2)$ one can assume the validity of (2.5) and (2.7) only for the very particular case of $n = 1$! We will now demonstrate that the last term in (2.5) is indeed divergent at the pole [19].

Finiteness of the $\mathcal{O}(g_2)$ piece of (2.5) was demonstrated in [8] where the coefficient was found to be,

$$\frac{\Gamma^{my}_n}{n^2 - (m/y)^2} = -\frac{m/y}{n + (m/y)} G^{12}_{n;my}\,. \tag{2.8}$$

Here, $G_{n;my}$ denotes the tree-level inner product between single and double trace operators which was first computed in [4],

$$G^{12}_{n;my} = \frac{g_2}{\sqrt{J}}\sqrt{\frac{1-y}{y}}\frac{\sin^2(\pi n y)}{\pi^2 (n - (m/y))^2}. \tag{2.9}$$

We will also make abundant use of the radiative corrections to single-double correlator which was also obtained in [8],

$$\Gamma^{12}_{n;my} = \left(\left(\frac{m}{y}\right)^2 - n\frac{m}{y} + n^2 \right) G^{12}_{n;my}. \tag{2.10}$$

Let us now investigate $\mathcal{O}(g_2^2)$ part of (2.5). First of all, it is not hard to show that there is no divergence at $n = \pm m'/y'$ in the second sum of the second term. These internal poles are canceled out by zeros of the numerator. Similarly one can show that (2.5) is finite at the external pole $n = -m/y$. This is done at the end of this section. However we shall shortly demonstrate that the external pole at $n = +m/y$ give rise to a divergence [19] hence render the use of non-degenerate perturbation theory invalid for $n > 1$.[4]

To go further we need (in addition to matrix elements already computed in the literature) the $O(g_2^2)$ contributions to

$$G^{13}_{n;myz}\,, \qquad \text{and} \qquad \Gamma^{13}_{n;myz}$$

and $O(g_2)$ contributions to

$$G^{23}_{ny;my'z'}\,, \qquad \text{and} \qquad \Gamma^{23}_{ny;my'z'}\,.$$

Necessary computations are summarized in appendix D and the results are as follows,

$$\begin{aligned} G^{13}_{n;myz} &= \frac{g_2^2}{\pi^2 J}\sqrt{\frac{z\tilde{z}}{y}}\frac{1}{(n-k)^2}\Big((1-y)\sin^2(\pi n y) + y(\sin^2(\pi n z) + \sin^2(\pi n \tilde{z})) - \\ &\qquad - \frac{1}{2\pi(n-k)}(\sin(2\pi n y) + \sin(2\pi n z) + \sin(2\pi n \tilde{z}))\Big) \end{aligned} \tag{2.11}$$

$$\Gamma^{13}_{n;myz} = \lambda'(n^2 + k^2 - nk)G^{13}_{n;myz} + B^{13}_{n;myz}\,. \tag{2.12}$$

Here $k = m/y$ is the world-sheet momentum of the double-string state and we defined $\tilde{z} = 1 - y - z$.

Let us digress to underline an important detail. As we showed in appendix D, among the contributions to the radiative corrections to single-triple correlator there are contractible, semi-contractible and non-contractible Feynman diagrams (see appendix D for definition of contractibility in planar diagrams). The contributions of the first two are summarized in the first term above, whereas $B^{13}_{n;myz}$ denotes the non-contractible contribution,

$$B^{13}_{n;myz} = \frac{2g_2^2\lambda'}{\pi^3 J}\frac{1}{(n - m/y)}\sqrt{\frac{z\tilde{z}}{y}}\sin(\pi n z)\sin(\pi n \tilde{z})\sin(\pi n(1-y))\,. \tag{2.13}$$

Double-triple coefficients receive $\mathcal{O}(g_2)$ contributions only from disconnected diagrams where the 2-3 process is separated as 1-1 and 1-2. Therefore these require somewhat simpler

[4] For $n = 1$ it is impossible to satisfy the degeneracy condition $n = m/y$.

computations and the results are,

$$G^{23}_{ny';myz} = y'^{3/2} G^{12}_{n;my/y'} (\delta_{y',y+z} + \delta_{y',1-z}) + \frac{g_2}{\sqrt{J}} \delta_{mn} \delta_{yy'} \sqrt{(1-y)z\tilde{z}} \tag{2.14}$$

$$\Gamma^{23}_{ny';myz} = \frac{n^2}{y'^2} G^{23}_{ny';myz} + y'^{3/2} (\delta_{y',y+z} + \delta_{y',1-z}) \frac{m}{y} \left(\frac{m}{y} - \frac{n}{y'} \right) \tag{2.15}$$

$$= \left(\frac{n^2}{y'^2} - \frac{n}{y'} \frac{m}{y} + \frac{m^2}{y^2} \right) G^{23}_{ny';myz} \,. \tag{2.16}$$

We are now ready to compute the $\mathcal{O}(g_2^2)$ term in (2.5). First of all one shows the curious fact[5] that

$$\Gamma^{myz}_n = 0 \,. \tag{2.17}$$

Γ^{myz}_n is decomposed as,

$$\Gamma^{myz}_n = G^{myz;m'} \Gamma_{m';n} + G^{myz;m'y'} \Gamma_{m'y';n} + G^{myz;m'y'z'} \Gamma_{m'y'z';n} \,. \tag{2.18}$$

One can easily invert $3 \times \infty$ dimensional matrix G_{ij}, by solving the equation $G^{ik} G_{kj} = \delta^i_j$ order by order in g_2. To $\mathcal{O}(g_2^2)$ one finds,

$$G^{ij} = \begin{pmatrix} \delta_{mn} + G^{12}_{m;py''} G^{12}_{py'';n} & -G^{12}_{m;ny'} & \frac{1}{2} G^{12}_{m;py''} G^{23}_{py'';ny'z'} - \frac{1}{2} G^{13}_{m;ny'z'} \\ -G^{12}_{my;n} & \begin{array}{c} \delta_{mn} \delta_{y,y'} + G^{12}_{my;p} G^{12}_{p;ny'} + \\ + \frac{1}{2} G^{23}_{my;py''z''} G^{23}_{py''z'';ny'} \end{array} & -\frac{1}{2} G^{23}_{my;ny'z'} \\ \frac{1}{2} G^{23}_{myz;py''} G^{12}_{py'';n} - \frac{1}{2} G^{13}_{myz;n} & -\frac{1}{2} G^{23}_{myz;ny'} & \begin{array}{c} \frac{1}{4} \delta_{m,n} \delta_{y,y'} (\delta_{z,z'} + \delta_{z,\tilde{z}'}) + \\ + \frac{1}{4} G^{23}_{myz;py''} G^{23}_{py'';ny'z'} \end{array} \end{pmatrix} .$$

By the use of decomposition identities listed in appendix C, one can prove that (2.18) vanishes (see appendix C for details).

Let us now consider the last term in (2.5). A calculation similar to the one that leads to (2.8) gives,

$$\Gamma^{myz}_{py'} = \frac{1}{2} k'(k'-p) \frac{G^{12}_{p;my/y'}}{\sqrt{y'}} (\delta_{y',y+z} + \delta_{y',y+\tilde{z}}) \,, \tag{2.19}$$

where $k' = my'/y$. The other necessary ingredient, Γ^{py}_n was already computed in [8]

$$\Gamma^{py'}_n = k(k-n) G_{n;py'}$$

where $k = p/y'$. Inserting these expressions into (2.5), we get the whole coefficient in front of O^J_{myz} as,

$$I = \frac{1}{n^2 - (m/y)^2} \int_0^1 \mathrm{d}y' \sum_{p=-\infty}^{\infty} \frac{k'(k'-p) k(k-n)}{2\sqrt{y'}(n^2 - (p/y')^2)} G_{p;my/y'} G_{n;py'} (\delta_{y',y+z} + \delta_{y',y+\tilde{z}}) \,. \tag{2.20}$$

[5] which finds a natural explanation in the formulation of [19]

Despite the appearance of $n^2 - (p/y')^2$ in the denominator there is no divergence at $n = \pm p/y'$ because $G_{n;py'}$ in the numerator also vanishes at these intermediate poles. We will now show however that I *is divergent at* $n = m/y$ [19]. The residue of I at $n = m/y$ is,

$$\left(n - \frac{m}{y}\right) I \bigg|_{n=m/y} = -\frac{g_2^2}{4\pi^4 J}\sqrt{\frac{z\tilde{z}}{y}} \int_0^1 \mathrm{d}y' \frac{1}{y'}(\delta_{y',y+z} + \delta_{y',y+\tilde{z}}) \sin^2(\pi n y') \times$$
$$\times \sum_{p=-\infty}^{\infty} \frac{p}{y'} \frac{\sin^2(\pi p y/y')}{(n^2 - (p/y')^2)(n - p/y')^2} .$$

We emphasize that the infinite series in this expression is a prototype for the non-trivial sums that appear in the computations involving triple-trace BMN operators. We explain the computation of this one and other similar sums which will be necessary for the next section in appendix B.[6] The result is,

$$\sum_{p=-\infty}^{\infty} \frac{p}{y'} \frac{\sin^2(\pi p y/y')}{(n^2 - (p/y')^2)(n - p/y')^2} = y' \left(\frac{\pi^3 y^2}{2} \cot(\pi n y') - \frac{\pi^2 y}{4n}\right) . \tag{2.21}$$

Inserting this in the above expression for the residue and evaluating the integral gives,

$$\left(n - \frac{m}{y}\right) I \bigg|_{n=m/y} = \frac{g_2^2}{J} \frac{y}{8n\pi^2} \sqrt{\frac{z\tilde{z}}{y}} \sin(\pi n z) \sin(\pi n \tilde{z}) .$$

Since the residue does not vanish at $n = +m/y$, (2.5) becomes divergent at this pole.

To see that there is no further divergence in (2.5) let us consider what happens at the other pole $n = -m/y$. It is easy to see from (2.8) that the second term is finite because G^{12} in the numerator linearly goes to zero as well as the denominator. The complicated second piece in the last term of (2.5) seems to be divergent at the first sight. Let us look at the residue at this pole,

$$\left(n - \frac{m}{y}\right) I_2 \bigg|_{n=-m/y} = -\frac{g_2^2}{4\pi^4 J}\sqrt{\frac{z\tilde{z}}{y}} \int_0^1 \mathrm{d}y' \frac{1}{y'}(\delta_{y',y+z} + \delta_{y',y+\tilde{z}}) \sin^2\left(\frac{\pi m y'}{y}\right) \times$$
$$\times \sum_{p=-\infty}^{\infty} \frac{p}{y'} \frac{\sin^2(\pi p y/y')}{((m/y)^2 - (p/y')^2)^2} .$$

This sum vanishes thanks to the antisymmetry of the summand. Thus we saw that all of the terms in (2.5) is finite at the pole $n = -m/y$.[7]

The fact that $I = \infty$ at $n = m/y$, hence (2.5) is ill-defined at this pole hints that *one should rather use degenerate perturbation theory to handle the diagonalization problem* [19]. Although somewhat disappointing, this result is by no means unexpected. On the GT side one can reason as follows.[8] 't Hooft limit suggests that anomalous dimensions of

[6] Unfortunately none of the well-known symbolic computation programs is helpful.

[7] Of course one has to worry about finiteness of (2.7) at $n = \pm m/y$. But this requires much less effort to see from the expressions for Γ^{23} and Γ^{12}.

[8] This is a suggestive argument due to Dan Freedman.

observables be expanded in powers of g_2^2 where $g_2 = 1/N$ in 't Hooft limit and $g_2 = J^2/N$ in the BMN double scaling limit. Had single-trace BMN operators been degenerate with double-trace BMN operators one would expect an $\mathcal{O}(g_2)$ shift in the single-trace anomalous dimension. This would be unexpected for the 't Hooft expansion of the observables hence might have indicated an inconsistency in the BMN theory. However the degeneracy of sigle and triple-trace BMN operators, at most, gives rise to an $\mathcal{O}(g_2^2)$ shift in Δ_1 which is not inadmissible. By the same token, one generally expects degeneracy among BMN operators with only odd-numbered traces and only even-numbered traces seperately. At order g_2^2, this result indeed follows by the scaling law of GT correlators derived in the next section provided that there is no degeneracy between single and double-traces and there is degeneracy between single and triple traces. Hence, at this order the degenerate subspace of BMN operators are divided into two subspaces which include odd and even numbered traces seperately.

On the string theory side the degeneracy of single and triple-trace operators indicate that a single-string state can decay into a triple-string state with the same world-sheet momentum $m = ny$. Furthermore, as we mentioned in the introduction, correspondence of trace number in GT and string number in ST loses its meaning for finite g_2. Therefore the general conclusion is that an initial string state that is composed of states of different string number but all on the same "momenta-shell", $n = m/y$, is generically unstable and can decay into states that are stable at $O(g_s^2)$. These stable states should be in correspondence with the eigen-operators of the degenerate subspaces in GT side. This conclusion is hardly surprising.

We shall not pursue this degeneracy problem further in this paper, but based on some preliminary calculations we make the following conjecture. Consider the degeneracy problem for finite J (which we choose as an odd number for convenience). Then two degenerate subspaces involve 1,3,...,J-trace and 2,4,...,J-1-trace operators seperately. To find whether degeneracy gives rise to a shift in the eigenvalues one should diagonalize the order g_2^2 "transition matrix", M_i^{i+2} (finite J version of (2.1)) at $n = m/y$ seperately for odd and even i. We conjecture that regardless the exact form of M, there exist an eigenstate O_1' that tend to the BMN operator $\tilde{O}_1$ (at order g_2^2) as $J \to \infty$ where $\tilde{O}_1$ is the mostly single-trace eigen-operator of the dilatation generator that is obtained by the non-degenerate formulation at $\mathcal{O}(g_2)$. Futhermore there exist i-trace eigenstates, O_i' in the degenerate subspace whose eigenvalues tend to the anomalous dimensions of $\tilde{O}_i$ that are obtained by naively using the non-degenerate formulation. Therefore the eigenvalues of these particular O_i' will coincide with the anomalous dimensions which can be obtained via non-degenerate theory ignoring the aforementioned mixing. For the case of $i = 1$ this can be understood as a justification of (1.5). For $i > 1$ this leads to a simplification in determination of the higher-trace anomalous dimensions which we employ in section 4. We prefer to leave this assertion as a conjecture in this paper.

In section 4 we will use this conjecture to make predictions about string-theory amplitudes. We will first compute the anomalous dimension of a general i-trace BMN operator, $\Delta_i - J$ by the method of non-degenerate perturbation theory. Since dilatation generator is supposed to correspond to P^-, this will give a prediction for the eigenvalue of P^- in the two-string sector. Next, we will move on to compute the modified mixing matrices, $\tilde{\Gamma}^{22}$ and $\tilde{\Gamma}^{13}$ in the string basis. This will allow us to make predictions for the corresponding matrix elements of P^-.

3. Dependence on g_2 and J of an arbitrary gauge theory correlator

As shown by detailed investigation in the recent literature n-point functions of the observables in the BMN limit come with definite dependence on the dimensionless parameters, λ' and g_2.[9] Generally, the correlators also have an explicit dependence on J which will turn out to be crucial in drawing conclusions about the corresponding string processes.

In this section we will discuss the g_2 and J dependence of a connected two-point correlator of multi-trace BMN operators with two scalar impurities:

$$C^{ij} \equiv \langle : \bar{O}_n^{J_1} \bar{O}^{J_2} \cdots \bar{O}^{J_i}(x) :: O_m^{J_{i+1}} O^{J_{i+2}} \cdots O^{J_{i+j}}(0) : \rangle \Big|_{\text{connected}} . \tag{3.1}$$

This should be clear from the discussion in appendix A.

The space-time dependence of (3.1) is trivial: $(4\pi^2 x^2)^{-J-2}$ in free theory and $(4\pi^2x^2)^{-J-2}\ln(x^2\Lambda^2)/(8\pi^2)$ at $\mathcal{O}(\lambda')$ where J is the total number of Z fields, i.e. $J = J_1 + \cdots + J_i = J_{i+1} + \cdots + J_{i+j}$ in (3.1). Without loss of generality, one can assume $j \leq i$. There are various connected and disconnected diagrams with different topology that contribute to (3.1). Since the disconnected contributions will be given by (3.1) for smaller i and j, it suffices to consider the fully-connected contribution to (3.1). In appendix A we prove that the fully-connected piece of (3.1) has the following general form,

$$C^{ij} = \frac{g_2^{i+j-2}}{J^{(i+j)/2-1}} \left\{ G^{ij} \frac{1}{(4\pi^2x^2)^{J+2}} - \lambda' \Gamma^{ij} \frac{1}{(4\pi^2x^2)^{J+2}} \ln(x^2\Lambda^2) \right\} . \tag{3.2}$$

Here the "free" and "anomalous" matrix elements, G and Γ are functions of world-sheet momenta, m, n and of the ratios J_s/J for $s = 1, \ldots i+j$.[10] One directly sees from (3.2) that the disconnected contributions to the same correlator are less supressed. To demonstrate this in a simple example consider the disconnected contraction where $\bar{O}_i$ and O_j are seperated into $\bar{O}_{i_1}$, $\bar{O}_{i_2}$ and O_{j_1}, O_{j_2} with $i_1 + i_2 = i$ and $j_1 + j_2 = j$ respectively and suppose $\bar{O}_{i_1}$ connects to O_{j_1}, $\bar{O}_{i_2}$ connects to O_{j_2}. The lowest order scaling law of this contribution is given by,

$$\frac{g_2^{i_1+j_1-2}}{J^{(i_1+j_1)/2-1}} \frac{g_2^{i_2+j_2-2}}{J^{(i_2+j_2)/2-1}} = \frac{g_2^{i+j-4}}{J^{(i+j)/2-2}} .$$

To compare this with the fully connected result in (3.2) consider one-loop correction to it which provides a factor of g_2^2 and which gives $\frac{g_2^{i+j-2}}{J^{(i+j)/2-2}}$. Clearly this disconnected

[9]For finite g_2 proof exist only at linear order in λ' although it is very likely to hold at higher loops. For $g_2 = 0$, [5] showed that sum of radiative corrections to single-single BMN correlator at all orders in $\mathcal{O}(g_{\mathrm{YM}}^2)$ can be expressed as a function only λ'.

[10]Note that the scaling law in front of the curly brackets represent the lowest order contribution in $g_2/\sqrt{J}$ to the fully connected $i-j$ correlator. In general there will be loop corrections to G and Γ in this expression which will modify the scaling law by the multiplicative factor g_2^{2n} where n is the number of loops.

JHEP10(2003)027

contribution is enhanced by a factor of J with respect to the fully connected one. Note that *suppression of C^{ij} in the BMN limit is absent only when $i = j = 1$.* We state the conclusion as,

> *Connected contributions to C^{ij} are suppressed in the BMN limit for i and/or j larger than 1.*

Let us briefly discuss the case of BPS type multi-trace operators. A BPS type multi-trace operator which involve two scalar type impurities is defined as,

$$O^J_{i;\phi\psi} =: O^{J_1}_\phi O^{J_2}_\psi O^{J_3} \cdots O^{J_i} :, \tag{3.3}$$

where,

$$O^J_\phi = \frac{1}{\sqrt{N^{J+1}}} \mathrm{Tr}(\phi Z^J) \tag{3.4}$$

and a similar definition for O^J_ψ.

One observes that a generic matrix element between BMN and BPS i-trace operators (both having the same number of traces) at all orders in g_2 are suppressed by a power of J. This can be seen by noting that these correlators should necessarily be partially connected (since both $O^{J_1'}_\psi$ and $O^{J_2'}_\phi$ in the BPS type i-trace operator should connect to the same $O^{J_1}_n$ in the i-trace BMN operator) hence suppressed by at least a factor of J with respect to BPS-BPS or BMN-BMN correlators of the same number of traces. Above scaling law tells us that, in the latter cases suppression at an arbitrary order in g_2 can only be avoided by completely disconnected graphs with an arbitrary number of loops. This simple observation allowed us to ignore BPS type double and triple operators in the previous section that would otherwise contribute in the intermediate sums.

4. Anomalous dimension of a general multi-trace BMN operator at $\mathcal{O}(g_2^2)$

The fact that disconnected contributions to the multi-trace correlators are suppressed as $J \to \infty$ has direct consequences for the scale dimension of multi-trace operators both at order g_2^2 and higher.

Call the i-trace BMN operator in (3.1) as O_i. Here i is a collection of labels, $i = \{n, y_1, \ldots, y_i\}$ with $y_1 \equiv J_1/J$, etc. Because of the non-vanishing mixing, $\langle \bar{O}_i O_j \rangle$ with multi-trace operators of different order ($i \neq j$), O_i is *not* an eigen-operator of $\Delta - J$ and a non-trivial diagonalization procedure is required to obtain the true scale dimension. Eigenvectors at $\mathcal{O}(g_2)$ is affected only by mixing of O_i with $O_{i\pm1}$. The diagonalization procedure is essentially equivalent[11] to non-degenerate perturbation theory [8] and as in section 2 one obtains the mostly i-trace eigen-operator as,

$$\tilde{O}_i = O_i + g_2 \sum_{j=i\pm1} \frac{\Gamma^j_i}{\rho_i - \rho_j} O_j \,, \tag{4.1}$$

[11] See the discussion at the end of section 2 for the effects of mixing with higher trace BMN operators.

where $\Gamma_i^j = G^{jk}\Gamma_{ki}$ and ρ_k is the $O(g_2^0)$ eigenvalue of the k-trace operator ($\rho_i = (n/y_1)^2$ in case of O_i in (3.1)).[12] To compute the eigenvalue we need, (using (4.1)),

$$\langle \bar{\tilde{O}}_i \tilde{O}_i \rangle = \langle \bar{O}_i O_i \rangle + 2g_2 \sum_{j=i\pm 1} \frac{\Gamma_i^j}{\rho_i - \rho_j} \langle \bar{O}_j O_i \rangle + g_2^2 \sum_{j,k=i\pm 1} \frac{\Gamma_i^j}{\rho_i - \rho_j} \frac{\Gamma_i^k}{\rho_i - \rho_k} \langle \bar{O}_j O_k \rangle \,. \tag{4.2}$$

Call the $\mathcal{O}(g_2^2)$ part of this quantity as,

$$\langle \bar{\tilde{O}}_i \tilde{O}_i \rangle|_{g_2^2} = \tilde{G}_{ii} - \lambda' \tilde{\Gamma}_{ii} \ln(x^2 \Lambda^2) \,. \tag{4.3}$$

Generalizing the method described in [8] to the case of multi-trace operators, we express the true scale dimension at this order in terms of the above quantities as,

$$\Delta_i|_{g_2^2} = \tilde{\Gamma}_{ii} - \rho_i \tilde{G}_{ii} \,. \tag{4.4}$$

It is not hard to see that (4.4) is equivalent to (2.3) as it should be. To compute Δ_i from either (4.4) or (2.3) one needs G_{ij} and Γ_{ij} to the necessary order. Although second method is a short-cut we prefer to start from (4.4) because this method makes it clear that modified operators, $\tilde{O}_i$ are true eigen-operators of the dilatation generator.

Now, the crucial point is to recall that the connected contributions to $\tilde{\Gamma}_{ii}$ and $\tilde{G}_{ii}$ are suppressed as $J \to \infty$ and the evaluation of these quantities reduce to the evaluation of only the fully disconnected pieces. For example the quantity G_{ii} receives non-zero contributions only from the following completely disconnected Wick contractions,

$$\begin{aligned} G_{ii}|_{g_2^2} \;=\; \langle \bar{O}_i O_i \rangle_{g_2^2} &= \langle \bar{O}_n^{J_1} O_n^{J_1} \rangle_{g_2^2} \langle \bar{O}^{J_2} O^{J_2} \rangle_{g_2^0} \cdots \langle \bar{O}^{J_i} O^{J_i} \rangle_{g_2^0} + \\ &+ \langle \bar{O}_n^{J_1} O_n^{J_1} \rangle_{g_2^0} \langle \bar{O}^{J_2} O^{J_2} \rangle_{g_2^2} \cdots \langle \bar{O}^{J_i} O^{J_i} \rangle_{g_2^0} + \cdots + \\ &+ \langle \bar{O}_n^{J_1} O_n^{J_1} \rangle_{g_2^0} \langle \bar{O}^{J_2} O^{J_2} \rangle_{g_2^0} \cdots \langle \bar{O}^{J_i} O^{J_i} \rangle_{g_2^2} \,. \end{aligned} \tag{4.5}$$

Now, we shall compute Δ_i at $\mathcal{O}(g_2^2)$ for arbitrary i. One first observes that $j = i-1$ channel in (4.2) necessarily gives connected diagrams hence suppressed by the power of J given by (3.2). More explicitly, the summands in the $(i-1)$ channel are $\mathcal{O}(g_2^2/J)$ but the intermediate sums do not provide a compensating factor of J unlike in the $(i+1)$ channel. This is illustrated in figure 1 in case of $i = 2$.[13] Similarly the summands in $(i+1)$ channel are also $\mathcal{O}(g_2^2/J)$, therefore only disconnected $i \to (i+1) \to i$ processes can contribute. This is also illustrated in figure 1. The conclusion is that,

1. i-$(i-1)$ mixing does not affect i-trace eigenvalue,

2. Only disconnected $i \to (i+1) \to i$ processes in i-$(i+1)$ mixing contribute. (see figure 2)

[12]The proof of the validity of non-degenerate perturbation theory at $\mathcal{O}(g_2^2)$ is illustrated in case of single-trace and double-trace operators in section 2. However this proof immediately generalizes to the general case of i-trace operators because of the suppression of connected correlators. Requirement of disconnectedness boils down the required computation to the one presented in section 2.

[13]For an explanation for these “string-like” transition diagrams see the end of this section.

JHEP10(2003)027

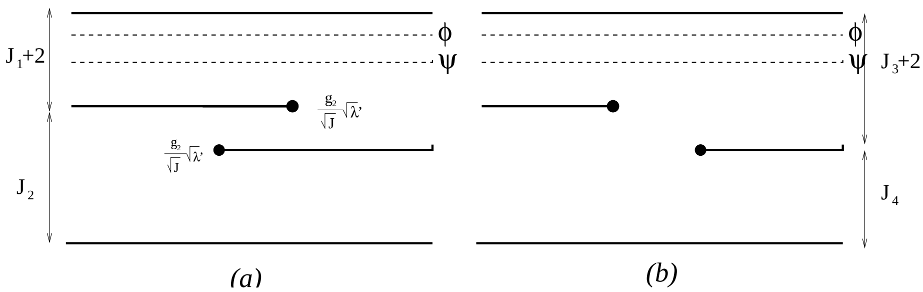

Figure 1: Left figure shows that connected contribution to $2 \to 3 \to 2$ process is suppressed by $1/J$. Right figure shows similar suppression of $2 \to 1 \to 2$ channel.

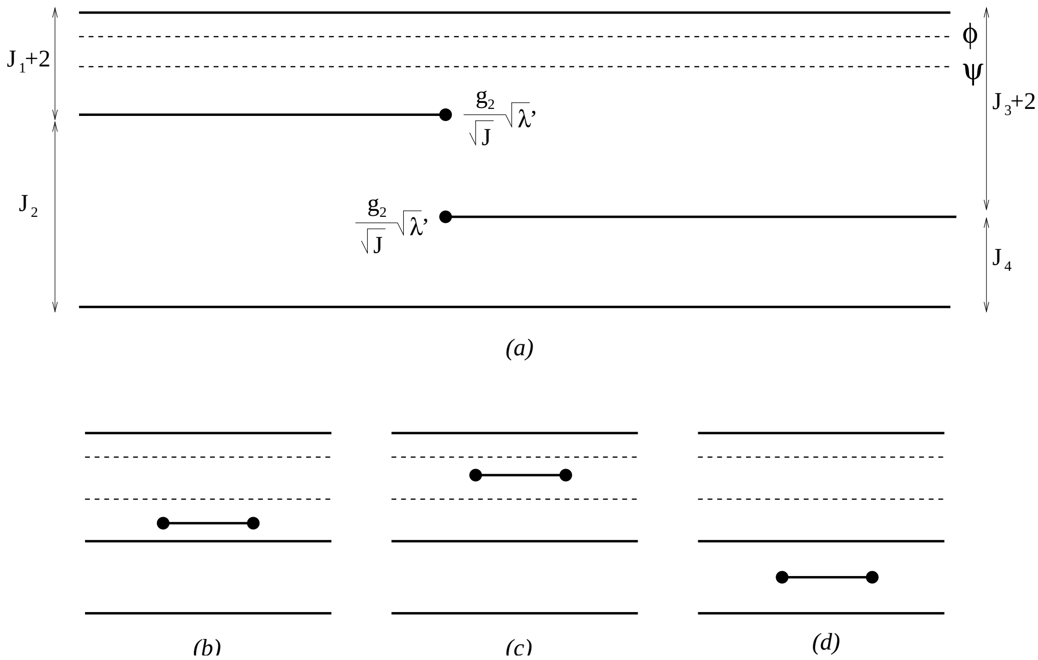

Figure 2: A representation of planar contributions to $\langle : \bar{O}_n^{J_1} \bar{O}^{J_2} :: O_m^{J_3} O^{J_4} : \rangle$ at $\mathcal{O}(g_2^2 \lambda')$. Dashed lines represent scalar impurities. We do not show Z lines explicitly. Vertices are of order $g_2\sqrt{\lambda'/J}$. (a) Connected contribution. (b), (c), (d) Various disconnected contributions.

One obtains the quantities, $\tilde{G}_{ii}$ and $\tilde{\Gamma}_{ii}$ that are necessary to evaluate (4.4) from (4.2). The former reads,

$$\begin{aligned} \tilde{G}_{ii} &= G_{ii} + 2\sum_{i+1} \frac{\Gamma_i^{i+1}}{\rho_i - \rho_{i+1}} G_{i+1,i} + \sum_{i+1,i'+1} \frac{\Gamma_i^{i+1}}{\rho_i - \rho_{i+1}} \frac{\Gamma_i^{i'+1}}{\rho_i - \rho_{i'+1}} G_{i+1,i'+1} \\ &= G_{ii} + \sum_{i+1} \frac{\Gamma_i^{i+1}}{\rho_i - \rho_{i+1}} \left(2G_{i+1,i} + i! \frac{\Gamma_i^{i+1}}{\rho_i - \rho_{i+1}} \right). \end{aligned} \tag{4.6}$$

Here, we use the indices in a schematic sense, for example $(i'+1)$ and $(i+1)$ are independent indices that both refer to a collective index which labels an $(i+1)$-trace operator, i.e. $i+1 = \{m, y_1, \ldots, y_i\}$ and $i'+1 = \{m', y'_1, \ldots, y'_i\}$. In the second line above, we used the expression for the lowest order, free two-point function of $(i+1)$-trace BMN operators. For general i, this is easily obtained by recalling the fact that only disconnected pieces contribute. Thus to lowest order, $\mathcal{O}(g_2^0)$, G_{ii} is product of its disconnected pieces summed over all ways of Wick contracting various BPS operators:

$$G_{ii'} = G_{my_1\cdots y_i;m'y'_1\cdots y'_i} = \delta_{mm'}\delta_{y_1y'_1}\sum_P \delta_{y_2y'_{P(2)}}\cdots\delta_{y_iy'_{P(i)}}\,, \tag{4.7}$$

where P runs over all permutations of the set $\{2,\dots i\}$. We stress that we need the $\mathcal{O}(g_2^2)$ expression for G_{ii} in (4.6) rather than (4.7). Using this formula for $G_{i+1,i'+1}$ in the first line of (4.6) and summing over the indices $i'+1$ produces a factor of $i!$. To go further we need

$$\frac{\Gamma_i^{i+1}}{\rho_i-\rho_{i+1}}\,.$$

The only $\mathcal{O}(g_2)$ contributions to the matrix element come from the following two terms,

$$\Gamma_i^{i+1} = G^{i+1,j+1}\Gamma_{j+1,i} + G^{i+1,j}\Gamma_{j,i}\,.$$

We need to invert the $2\times\infty$ dimensional matrix of inner products between i-trace and $(i+1)$-trace operators.[14] It is not hard to find the inverse perturbatively at $\mathcal{O}(g_2)$ with the result,

$$G^{A,B} = \begin{pmatrix} \frac{G_{i,i'}}{(i-1)!(i-1)!}+\mathcal{O}(g_2^2) & -\frac{G_{i,i'+1}}{(i-1)!i!}+\mathcal{O}(g_2^3) \\ -\frac{G_{i+1,i'}}{(i-1)!i!}+\mathcal{O}(g_2^3) & \frac{G_{i+1,i'+1}}{i!i!}+\mathcal{O}(g_2^2) \end{pmatrix}.$$

We also need *free* i-$(i+1)$ correlator at $\mathcal{O}(g_2)$. Since it is given by the fully-disconnected contribution, it is obtained as a simple generalization of (2.14):

$$\begin{aligned} G^{i,i+1}_{my_1\cdots y_i;m'y'_1\cdots y'_{i+1}} &= \delta_{m,m'}\delta_{y_1,y'_1}\frac{g_2}{\sqrt{J}}\sum_{P,P'}\delta_{y_{P(2)},y'_{P'(2)}}\cdots\delta_{y_{P(i-1)},y'_{P'(i-1)}}\delta_{y_{P(i)},y'_{P'(i)}+y'_{P'(i+1)}}\times \\ &\quad\times\sqrt{(1-y_{P(i)})y'_{P'(i)}y'_{P'(i+1)}}\,+ \\ &\quad+y_1^{3/2}G^{12}_{m,m'y'_1/y_1}\sum_P\delta_{y_2,y'_{P(2)}}\cdots\delta_{y_i,y'_{P(i)}}\delta_{y_1,y'_1+y'_{P(i+1)}}\,. \end{aligned} \tag{4.8}$$

Here the first term is a generalization of the second term in (2.14) and the second term is the generalization of the first term in (2.14). The sum P in the first term is over cyclic permutations of the set $\{2,\dots,i\}$ i.e. it has dimension $i-1$ and sum P' is over all possible ways of choosing two indices out of the set $1,\dots i+1$ (to form the single-double BPS correlator with $P(i)$th BPS operator in O_i) and than taking all possible permutations in the rest of the indices, i.e. $\dim(P') = (i-2)!i(i-1)/2$.

[14] See section 3 for a justification of our omitting BPS type i-trace and $(i+1)$-trace operators in the evaluation of the eigenvalue.

JHEP10(2003)027

Finally we need the first order *radiative corrections* to this correlator. Much as in (2.16) this is given as,

$$\begin{aligned}\Gamma_{i,i+1} = \Gamma_{my_1\cdots y_i;m'y'_1\cdots y'_{i+1}} &= \frac{m^2}{y_1^2} G^{i,i+1}_{my_1\cdots y_i;m'y'_1\cdots y'_{i+1}} + y_1^{3/2} G^{12}_{m,m'y'_1/y_1} \frac{m'}{y'_1}\left(\frac{m'}{y'_1} - \frac{m}{y_1}\right) \times \\ &\quad \times \left\{\sum_P \delta_{y_2,y'_{P(2)}} \cdots \delta_{y_i,y'_{P(i)}} \delta_{y_1,y'_1+y'_{P(i+1)}}\right\} \\ &= \left(\frac{m^2}{y_1^2} - \frac{mm'}{y_1 y'_1} + \frac{m'^2}{y'^2_1}\right) G_{my_1\cdots y_i;m'y'_1\cdots y'_{i+1}} \,. \end{aligned} \tag{4.9}$$

Eqs. (4.8) and (4.9) are sufficient to determine Γ_i^{i+1} at $\mathcal{O}(g_2)$:

$$\begin{aligned}\Gamma_i^{i+1} &= G^{i+1,j+1}\Gamma_{j+1,i} + G^{i+1,j}\Gamma_{j,i} \\ &= \frac{1}{i!} y_1^{3/2} G^{12}_{m,m'y'_1/y_1} \frac{m'}{y'_1}\left(\frac{m'}{y'_1} - \frac{m}{y_1}\right)\left\{\sum_P \delta_{y_2,y'_{P(2)}} \cdots \delta_{y_i,y'_{P(i)}} \delta_{y_1,y'_1+y'_{P(i+1)}}\right\}. \end{aligned} \tag{4.10}$$

We insert this expression in (4.6) and perform the sum over the intermediate $(i+1)$ channel. Most of the terms in the contraction of delta-functions will be suppressed (e.g. first term in (4.8) multiplied with Γ_i^{i+1} and summed over $(i+1)$ is suppressed by $1/J$). Result is,

$$\tilde{G}_{ii'} = G_{ii'} - \delta_{ii'} \sum_{ny} y^3 G^{12}_{m,ny/y_1} G^{12}_{m',ny/y'_1} \frac{(\frac{n}{y})(\frac{n}{y} + \frac{m}{y_1} + \frac{m'}{y'_1})}{(\frac{n}{y} + \frac{m}{y_1})(\frac{n}{y} + \frac{m'}{y'_1})} \,. \tag{4.11}$$

Here, $\delta_{ii'}$ is a shorthand for the delta functions that arise from disconnected Wick contractions:

$$\delta_{ii'} = \delta_{y_1,y'_1} \sum_P \delta_{y_2,y'_{P(2)}} \cdots \delta_{y_i,y'_{P(i)}} \,.$$

A completely analogous computation yields $\tilde{\Gamma}_{ii'}$ as,

$$\tilde{\Gamma}_{ii'} = \Gamma_{ii'} - \delta_{ii'} \sum_{ny} y^3 G^{12}_{m,ny/y_1} G^{12}_{m',ny/y'_1} \frac{(\frac{n}{y})(\frac{n^3}{y^3} + \frac{m^3}{y_1^3} + \frac{m'^3}{y'^3_1})}{(\frac{n}{y} + \frac{m}{y_1})(\frac{n}{y} + \frac{m'}{y'_1})} \,. \tag{4.12}$$

For the same reason as above we only need disconnected contributions to G_{ii} and Γ_{ii} which are trivial to evaluate. In case of $i = 2$ required diagrams are illustrated in figure 2*b*, 2*c* and 2*d*. In terms of the known expression for single string correlator at $\mathcal{O}(g_2^2)$ and radiative corrections to this [4, 2], one readily gets (see (4.5))

$$G_{ii'} = g_2^2 \delta_{ii'} \left(y_1^4 A_{mm'} + \frac{\delta_{mm'}}{24} \sum_{r=2}^{i} y_r^4\right), \tag{4.13}$$

and

$$\Gamma_{ii'} = g_2^2 \delta_{ii'} \left(y_1^2 (m^2 - mm' + m'^2) A_{mm'} + y_1^2 B_{mm'} + \frac{\delta_{mm'}}{24} \frac{m^2}{y_1^2} \sum_{r=2}^{i} y_r^4\right), \tag{4.14}$$

where A and B matrices are first defined in [4] and are reproduced in eqs. (C.1) and (C.2).

Let us digress for a moment to discuss a simpler type of degeneracy in energy eigenvalues that is referred as *momenta-mixing*. So far, we formulated our discussion in terms of the multi-trace BMN operators as given in eqs. (1.3) and (1.2). In doing so we ignored a degeneracy in the energy eigenvalues corresponding to operators with opposite world-sheet momenta, namely O_n^J and O_{-n}^J carry the same energy that is n^2. To incorporate the effects of this momenta-mixing one should disentangle the degenerate states by going to $\pm$ basis,

$$O_n^{\pm\,J} = \frac{1}{\sqrt{2}}(O_n^J \pm O_{-n}^J)\,. \tag{4.15}$$

In $\pm$ basis BMN operators with two scalar impurities transform in the singlet and triplet representation under the SU(2) subgroup of the full SO(4) R-symmetry. One can easily reformulate our results in this basis. For example the eigenvalue equation reads,

$$\Delta^{\pm}_{my_1\ldots y_i\,;\,my_1\ldots y_i}\Big|_{g_2^2} = \Delta_{my_1\ldots y_i\,;\,my_1\ldots y_i}\Big|_{g_2^2} \pm \Delta_{my_1\ldots y_i\,;\,-my_1\ldots y_i}\Big|_{g_2^2} \tag{4.16}$$

Using eqs. (4.13), (4.14), (4.11) and (4.12), it is straightforward to see that,

$$\Delta_{my_1\ldots y_i\,;\,-my_1\ldots y_i}\Big|_{g_2^2} = 0\,. \tag{4.17}$$

Thus, our first observation is that

> *Scaling dimension of all multi-trace BMN operators remain degenerate in $\pm$ channels at $\mathcal{O}(g_2^2)$.*

This fact can be explained by relating the multi-trace BMN operators in $+$ and $-$ channels by a sequence of supersymmetry transformations [14, 8, 16]. We would also like to emphasize that degeneracy of multi-trace BMN operators can be viewed as a consistency check on our long computation because our results are valid also for single-trace BMN operators for $i = 1$, where this degeneracy is well-established [7, 8].

Having established the degeneracy in $\pm$ basis, we can compute the eigenvalue by using (4.13), (4.14), (4.11) and (4.12) in

$$\Delta_i^{\pm}\Big|_{g_2^2} = \tilde{\Gamma}_{ii} - \rho_i \tilde{G}_{ii}\,.$$

Straightforward computation gives,

$$\Delta_i^{\pm} = y_1^2 \left(g_2^2 B_{mm} - J \int_0^1 \mathrm{d}x \sum_{n=-\infty}^{\infty} (G^{12}_{m,nx})^2 \frac{k^2(k^2 - m^2)}{(k+m)^2} \right),$$

where $k = m/x$. Using (2.9) one gets,

$$\Delta_i^{\pm}\Big|_{g_2^2} = \frac{y_1^2}{4\pi^2} \left(\frac{1}{12} + \frac{35}{32\pi^2 m^2} \right). \tag{4.18}$$

This is exactly the single-trace anomalous dimension that were computed in [7] and [8] up to the normalization factor y_1^2. This is hardly surprising given the fact that all Feynman diagrams that contribute to the evaluation of Δ_i separate into completely disconnected pieces. Since the only piece that can contribute to anomalous dimension is coming from the single-trace BMN sub-correlator (BPS sub-correlators are protected), we obtain the single-trace anomalous dimension as a result. However, from a general point of view this is a striking result and is one of the main conclusions of this paper:

> *Scaling dimension of all multi-trace BMN operators are determined by the dimension of the single-trace operator as*
>
> $$\Delta_i^{\pm} = \left(\frac{J_1}{J}\right)^2 \Delta\,.$$

Let us make an important observation. Consider a single-trace BMN operator with momentum n, hence with zeroth order energy n^2 and a multi-trace BMN operator with a different momentum m but with the same zeroth order energy, i.e. $n^2 = m^2/y_1^2$. Then, our formula above shows that the zeroth order degeneracy in the energy spectrum is lifted at order g_2^2 since,

$$\Delta_n - \Delta_{my_1\cdots y_i} = \frac{1-y_1^2}{48\pi^2}$$

according to eqs. (4.18) and (1.5). This only vanishes when $y_1 = 1$, namely when i-trace operator is replaced with a single-trace operator.

We believe that this result will also hold at higher orders in g_2 because the fact that only disconnected Feynman diagrams survive the BMN limit is still valid for higher orders in genus expansion. We see this by noticing that each g_2 comes along with a factor of $1/\sqrt{J}$ in the expansion, (see eq. (3.2), also figure 2 below). This should become more clear in the following.

This result establishes a firm prediction for $\mathcal{O}(g_2^2)$ eigenvalues of the light-cone SFT hamiltonian. When translated into string language, this prediction reads,

$$\langle\psi_i|P^-|\psi_i\rangle = \left(\frac{p_1^+}{p^+}\right)^2 \langle\psi_1|P^-|\psi_1\rangle.$$

We would like to emphasize that this prediction is completely independent of the field theory basis which identifies operators that are dual to the string states.

Let us now discuss the implications of our findings for some of the string amplitudes. For this let us represent our discussion about the scaling of correlators with g_2 and J in a diagrammatic way that is suggestive for light-cone SFT. For instance we represent the double-trace correlator in the BMN basis, $\langle\bar{O}_2 O_2\rangle_{g_2^2\lambda'}$, as in figure 2 where figure 2a shows the connected contribution to this correlator while figure 2b, 2c, 2d, represent the disconnected contributions at this order. Here each vertex represent a factor of Γ^{12} which is defined as,

$$\langle\bar{O}_1(x)O_2(0)\rangle_{g_2^2} = -\Gamma^{12}\lambda'\frac{\ln(x^2\Lambda^2)}{(4\pi^2x^2)^{J+2}}\,.$$

This quantity was first computed in [8] and given in eqs. (2.10), (2.9) which show that each vertex scale with a factor of $g_2/\sqrt{J}$. It is now clear that one can reproduce all of the information contained in the scaling law of (3.2) by representing the correlators with these

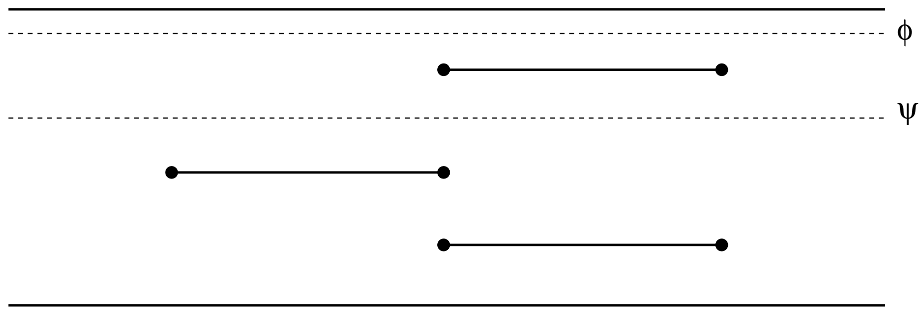

Figure 3: Connected contribution to double-triple correlator, $\langle \bar{O}_{ny} O_{myz} \rangle$, does give non-zero contribution in $1 \to 1$ process. For example this diagram will show up in the computation of $\mathcal{O}(g^6)$ scale dimension of single-trace operators.

diagrams. For instance the connected diagrams in figure $2a$ is $\mathcal{O}(g_2^2/J)$ hence vanishes in BMN limit whereas the disconnected diagrams of the same order in figure $2b$, $2c$ and d scale as g_2^2 therefore they are finite because of the extra J factor provided by the integration over the loop position.

To make contact with light-cone SFT we take this diagrammatic representation seriously with one qualification: The matrix elements of the light-cone hamiltonian should correspond to the matrix Γ in the *string basis*, not in BMN basis. As discussed in appendix B, this matrix element is obtained from Γ_{ij} with a unitary transformation, $\tilde{\Gamma} \equiv U \Gamma U^\dagger$. Whatever the correct identification of U is, this transformation will not change the scaling of Γ because it is independent of J. Therefore we can take the diagrams in figure 2 seriously as string theory diagrams.[15] where the vertices Γ replaced with $\tilde{\Gamma}$ which scale in the same way as before. For instance, the vanishing of figure $2a$ implies that there is no double-double "contact term" that contributes to $\langle \psi_2 | P^- | \psi_2 \rangle$ at $\mathcal{O}(g_s^2)$. This observation immediately generalize as,

> *There are no $O(g_s^2)$, i-i contact terms that contribute to $\langle \psi_i | P^- | \psi_i \rangle$.*

However, these contact terms do give contributions at higher orders in g_s^2. In other words, the suppression of the correlators in (3.2) does not imply the absence of physical information contained in these quantities. They certainly yield non-zero contributions to single-single loop corrections as illustrated in figure 3.

Let us also observe that the suppression of the diagram in figure $1b$ implies that there is also no $2 \to 1 \to 2$ contribution to this matrix element in string perturbation theory. This fact generalizes as,

> *String theory processes where the number of internal propagations is less than i, do not contribute $\langle \psi_i | P^- | \psi_i \rangle$.*

These assertions might seem strong, however one should note the important assumptions that were made in the above discussion. First of all the correspondence with GT, at

[15] To be precise, one should replace stripes in these 2D figures with tubes.

the perturbative level only holds when $\lambda' \ll 1$ which translates into the condition $\mu \gg 1$ in string theory. Therefore our discussion is valid for large values of μ. Secondly, the string amplitudes we consider involve a very particular class of external string states, namely the states with only two excitations along $i = 1, 2, 3, 4$ directions (corresponding to scalar impurities in the BMN operators). Note however that our discussion does not make any restriction to these particular two scalar excitations in the *internal string states*.

5. Matrix elements of pp-wave hamiltonian in 2-2 and 1-3 sectors

We will first compute the matrix elements of P^- in the two-string sector by the method of [13]. Assuming the validity of the basis transformation U_G that we discussed in appendix B, this will allow us to make a gauge theory prediction for SFT. Then, by the same method we will obtain the matrix elements in single-triple string sector. Let us briefly review the method.

In appendix B we presented a prescription to identify the string basis in field theory by transforming the basis of BMN observables with a real and symmetric transformation which renders the metric G_{ij} diagonal. The conjecture is that, matrix elements of P^- should be in correspondence with the matrix of $\mathcal{O}(\lambda')$ piece of the field theory correlators in the string basis. This is related to the same quantity in the old basis as,

$$\tilde{\Gamma} = U_G \Gamma U_G^\dagger .$$

We are interested in the $\mathcal{O}(g_2^2\lambda')$ piece of $\tilde{\Gamma}$. Using (B.2), this reads [13],

$$\tilde{\Gamma}^{(2)} = \Gamma^{(2)} - \frac{1}{2}\{G^{(2)}, \Gamma^{(0)}\} - \frac{1}{2}\{G^{(1)}, \Gamma^{(1)}\} + \frac{3}{8}\{(G^{(1)})^2, \Gamma^{(0)}\} + \frac{1}{4} G^{(1)}\Gamma^{(0)}G^{(1)}, \qquad (5.1)$$

where the superscript denotes the order in g_2. Straightforward algebra gives,

$$\begin{aligned}
\tilde{\Gamma}^{22}_{my;m'y'} = \Gamma^{22}_{my;my'} &- \frac{1}{2}\left(\left(\frac{m}{y}\right)^2 + \left(\frac{m'}{y'}\right)^2\right) G^{22}_{my;m'y'} - \\
&- \frac{1}{2}\left(G^{21}\Gamma^{12} + \Gamma^{21}G^{12} + G^{23}\Gamma^{32} + \Gamma^{23}G^{32}\right)_{my;m'y'} + \\
&+ \frac{3}{8}\left(G^{21}G^{12} + G^{23}G^{32}\right)_{my;m'y'}\left(\left(\frac{m}{y}\right)^2 + \left(\frac{m'}{y'}\right)^2\right) + \\
&+ \frac{1}{4}\left(G^{12}G^{12}n^2 + G^{23}\Gamma^{33}G^{32}\right)_{my;m'y'}
\end{aligned}$$

Γ^{22} and G^{22} in the first two terms are $\mathcal{O}(g_2^2)$ pieces of the corresponding matrices and Γ^{33} in the last term is the $\mathcal{O}(1)$ piece. Repeated intermediate indices mean summing over all possible operators that may appear in that intermediate process e.g. in the expression $G^{23}G^{32}$ one should sum over both BPS type *and* BMN type triple trace operators. Remarkable simplifications occur, when one recalls that only non-vanishing contributions in the double-double sector comes from disconnected diagrams. A term like $G^{21}G^{12}$ and $G^{21}\Gamma^{12}$ cannot be disconnected hence of $\mathcal{O}(1/J)$ and decouples in the BMN limit. Similarly one only keeps the disconnected contributions to G^{22} and Γ^{22}. All of the necessary ingredients to compute this expression except,

$$\Gamma^{33} = \begin{pmatrix} \frac{1}{2}(\frac{m}{y})^2 \delta_{m,m'}\delta_{y,y'}(\delta_{z,z'} + \delta_{z,1-y-z'}) & 0 \\ 0 & 0 \end{pmatrix},$$

were presented in section 2. This matrix tells us that there is no anomalous mixing among BPS type and between BMN and BPS type triple-trace operators at the zeroth order in g_2.

With the help of the decomposition identity (B.4), one obtains the matrix element of P^- in the double string-double string sector as,

$$\tilde{\Gamma}_{my;m'y'} = g_2^2 \delta_{y,y'} \frac{y^2}{4} B_{m,m'} . \tag{5.2}$$

For the definition of matrix B, see appendix C.

A similar calculation yields the single-triple matrix element in the string basis as,

$$\begin{aligned} \tilde{\Gamma}^{13}_{m;m'y'z'} &= \Gamma^{13}_{m;my'z'} - \frac{1}{2}\left(m^2 G^{13} + G^{13}\Gamma^{33}\right)_{m;m'y'z'} - \frac{1}{2}\left(G^{12}\Gamma^{23} + \Gamma^{12}G^{23}\right)_{m;m'y'z'} + \\ &\quad + \frac{3}{8}\left(\Gamma^{11}G^{12}G^{23} + G^{12}G^{23}\Gamma^{33}\right)_{m;m'y'z'} + \frac{1}{4} G^{12}_{m;ny''}\left(\frac{n}{y''}\right)^2 G^{23}_{ny'';m'y'z'} . \end{aligned}$$

Again, repeated indices in the intermediate sums imply the inclusion of all possible operators of that given type. For instance in the term $G^{12}G^{23}\Gamma^{33}$, one should use both BMN and BPS type double and triple operators in the intermediate process. A simplification occurs however when one notes that there is no $\mathcal{O}(g_2)$ contribution to G^{23} and Γ^{23} for a BPS type double-trace operator, the lowest order non-zero contribution appearing at $\mathcal{O}(g_2^3)$. By repeated use of the decomposition identities (B.5), (C.12) and (C.13) one obtains the amazingly simple expression,

$$\tilde{\Gamma}^{13}_{m;m'y'z'} = \frac{g_2^2}{4} B^{13}_{m;m'y'z'} .$$

Here B^{13} is the contribution to Γ^{13} from non-nearest neighbour interactions, given by (2.13). Thus we obtain the following GT prediction for the matrix elements of P^- in 1 string-3 string sector:

$$\tilde{\Gamma}^{13}_{n;myz} = \frac{g_2^2 \lambda'}{2\pi^3 J} \frac{1}{(n - m/y)} \sqrt{\frac{z\tilde{z}}{y}} \sin(\pi n z) \sin(\pi n \tilde{z}) \sin(\pi n(1-y)) . \tag{5.3}$$

Some comments are in order. First of all we note the striking similarity of $\tilde{\Gamma}^{22}$ and $\tilde{\Gamma}^{13}$ to $\tilde{\Gamma}^{11}$ that was obtained in [9, 13]:

$$\tilde{\Gamma}^{11}_{m;m'} = \frac{g_2^2}{4} B_{m,m'} .$$

In the 2-2 sector this just follows from the disconnectedness of the GT diagrams, hence the 2-2 matrix element just reduces to 1-1 case up to an overall factor y^2 therefore is hardly surprising. But our result for the 1-3 matrix elements indicates the following generalization. As first computed by Vaman and Verlinde [10] using SBF and then by Roiban, Spradlin and Volovich [12] using rigorous SFT the matrix element $\tilde{\Gamma}^{11}$ represents the "contact term" i.e. the $\mathcal{O}(g_s^2)$ matrix element of P^- between two single-string states in the ST side. On the GT side, in all of the cases we considered this matrix element is determined solely by the "non-contractible" contribution to Γ^{ij}. It is tempting to conjecture that the "non-contractible" GT diagram encodes the information for the $\mathcal{O}(g_s^2)$ contact term in PP-wave SFT. To check this conjecture one should compute $\mathcal{O}(g_s^2)$ matrix element of P^- between a single and a triple-string state and compare with (5.3).

6. Discussion and outlook

There are three main results in this manuscript. First of all, we demonstrated that non-degenerate perturbation theory becomes invalid at order g_2^2, as single-trace operators are degenerate with triple-trace operators. This result casts some doubt on the previously computed anomalous dimensions in [8, 7]. However we conjectured that some particular eigenstates of the degenerate subspace for finite J, tend to the modified BMN operators $\tilde{O}_i$ in the BMN limit whose eigenvalues coincide with the dimensions of $\tilde{O}_i$. Therefore the use of non-degenerate perturbation theory can be justified for these particular dilatation eigen-operators. This problem requires further investigation and it will be interesting to explore new effects related to this degeneracy problem in future.

Our second result is the determination of the anomalous dimensions of all multi-trace BMN operators that include two scalar impurity fields in terms of the single-trace anomalous dimension. We proved this interesting result to order $g_2^2\lambda'$ but the fact that connected field theory diagrams are suppressed also at higher orders in g_2 suggests that the conclusion holds at an arbitrary level in perturbation theory. (Of course, one has to first establish the validity of perturbation theory at higher orders.) These predictions for the eigenvalues of P^- are basis independent and therefore provide a firm prediction for SFT. It would be interesting to understand the string theory mechanism that is analogous to the BMN suppression that leads to vanishing of connected field theory diagrams. A natural next step in this analysis is to consider the anomalous dimensions of BMN operators that include higher number of impurities. We believe that suppression of the disconnected GT diagrams will lead to remarkable simplifications also in that problem.

Finally, we obtained predictions for the matrix elements of the light-cone hamiltonian in 2-2 and 1-3 string sectors. We emphasize that these predictions are sensitive to the way the string basis in GT is identified, unlike the predictions of section 3 for the *eigenvalues* of P^-. We fixed the basis with the assumption that the form of the basis transformation at $\mathcal{O}(g_2)$ is also valid at $\mathcal{O}(g_2^2)$. Although this assumption passed a non-trivial test in predicting the correct $\mathcal{O}(g_2^2)$ contact term of single-string states [12], there is no obvious reason to believe its validity for instance in the single-triple sector. Thus, our predictions can also be used as a test of the basis identifications of either [9] or [13] which are equivalent to each other at $\mathcal{O}(g_2^2)$.

Acknowledgments

It is a pleasure to thank Neil Constable, Dan Freedman, Matt Headrick, Charlotte Kristjansen, Mark Spradlin and Anastasia Volovich for useful discussions. This work is supported by funds provided by the D.O.E. under cooperative research agreement #DF-FC02-94ER40818.

A. Disconnectedness of GT correlators

In this appendix we will derive eq. (3.2). A study of the corresponding Feynman diagrams suffice to obtain the leading order scaling of a generic correlator with g_2 and J. Dependence on g_2 of a correlator is fixed by power of N in a Feynman diagram. This can be determined either by direct evaluation of the traces over the color structure (all fields are in adjoint rep. in $\mathcal{N}=4$ SYM) or by loop-counting. Since we are interested in the leading order g_2 dependence, the latter is easier. Explicit J dependence is determined by working out the symmetry factors in a Feynman diagram. As a warm-up consider the *free* extremal correlation function,

$$C^{i,1} = \langle \bar{O}^J_n : O^{J_1}_m O^{J_2} \cdots O^{J_i} :\rangle \,.$$

Leading order diagram drawn on a plane is shown in figure 4. Taking the normalization factor $\sim 1/N^{J+2}$ into account, trivial loop counting teaches us that,

$$C^{i,1} \propto \frac{1}{N^{i-1}} = \frac{g_2^{i-1}}{J^{2i-2}} \,. \tag{A.1}$$

Now, consider the combinatorics in figure 4 to determine the power of J. Planarity requires Wick contraction of O^{J_i}'s into $\bar{O}^J_n$ as a whole. Fix the position of, say O^{J_2} in $\bar{O}^J_n$. Then one has to sum over positions of other O^{J_i} operators for $i>2$ within $\bar{O}^J_n$ obtaining a factor of J^{i-2}. There is a phase summation over positions of ϕ and ψ impurities in $\bar{O}^J_n$, giving a factor of J^2. Cyclicity of O^{J_i}, $i>1$, provides a factor of J^{i-1}. Taking into account the $J^{-(i+1)/2}$ suppression from the normalization and $\mathcal{O}(J^{-2i+2})$ suppression in (A.1), we learn that,

$$C^{i,1} \sim \frac{g_2^{i-1}}{J^{(i-1)/2}} \,. \tag{A.2}$$

Next task is to obtain similar information for a general, non-extremal *free* correlator in (3.1). Without loss of generality, one can assume $j \leq i$. There are various connected and disconnected diagrams with different topology. It suffices to consider the fully connected contribution as explained in section 3. We first seek for the dependence on g_2 of the leading order (planar) fully connected diagram. As an example, a list of all distinct topologies for fully connected $i=3$, $j=2$ correlator is shown in figure 5. It is immediate to see that conservation of number of legs for each operator in the correlator (for each node in figure 5) requires that *all planar fully-connected diagrams have same g_2 power* irrespective of the topology (here, by topology we refer to different type of diagrams that are exemplified in figure 5, not the order in g_2). Then, it is sufficient to count the loops in a connected diagram that is the simplest for loop counting purposes. This simplest diagram is shown

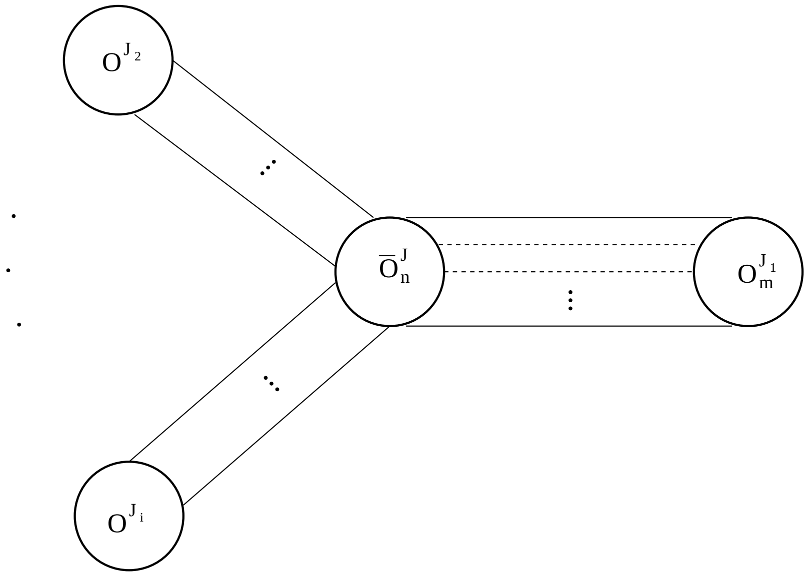

Figure 4: A typical planar contribution to $C^{1,i}$. Circles represent single-trace operators. Dashed lines denote impurity fields. Z lines are not shown explicitly and represented by "$\cdots$".

in figure 6. Each outer leg in figure 6 represent a bunch of J_s propagators (inner line has $J_1 - J_{i+2} - \cdots - J_{i+j} + 2$ propagators). Drawn on a plane, this means that there are a total of $(J+2) - (i+j-1) + 1$ loops in figure 6, including the circumference loop. Finally, a factor of N^{J+2} from normalizations and we obtain that,

$$C^{i,j} \propto \frac{1}{N^{i+j-2}} = \frac{g_2^{i+j-2}}{J^{2(i+j-2)}} . \tag{A.3}$$

Apart from the dependence on J coming from singling out the g_2 dependence as above, there are additional contributions from the combinatorics and normalizations. Determination of the power of J from the combinatorics works much as in the case of $C^{1,i}$. Fixing position of one O^{J_i} inside another operator that it connects to, we are left with sum over position of $i+j-3$ operators. This reasoning holds only for tree-type diagrams like in the first diagrams shown in figure 5a and figure 5b.

But it is not hard to see that the combinatorial factor for diagrams involving loops e.g. second diagram in figure 5. is also equal to $i+j-3$. This is because for each factor that one loses from the sum over positions of O_{J_i} because of the appearance of a loop, one gains a compensating factor of J for the loop summation. Also, cyclicity of BPS type operators within (3.1) provides a factor of J^{i+j-2}. Finally, including the powers of J coming from the normalizations and (A.3) we arrive at the general result,

$$C^{i,j} \sim \frac{g_2^{i+j-2}}{J^{(i+j)/2-1}} . \tag{A.4}$$

Note that this is for the connected contribution to (3.1). To obtain the g_2 and J dependence of disconnected contributions, one simply uses (A.4) for smaller i and/or j which shows that disconnected diagrams have lower powers in g_2 and they are less suppressed by a power of J.

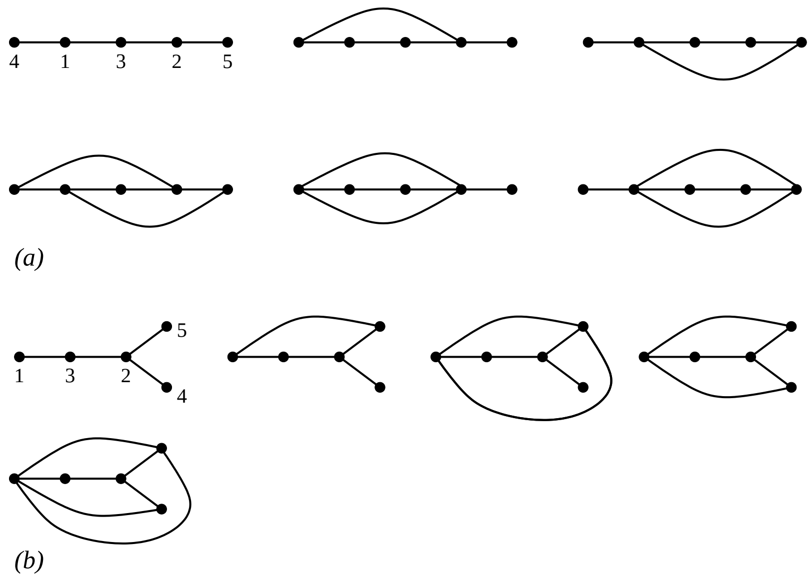

Figure 5: All distinct topologies of planar Feynman diagrams that contribute to $\langle : \bar{O}^{J_1}_n \bar{O}^{J_2} :: O^{J_3}_m O^{J_4} O^{J_5} : \rangle$. Nodes represent the operators while solid lines represent a bunch of Z propagators. Line between the nodes 1 and 3 also include two scalar impurities ϕ and ψ. All other topologies are obtained from these two classes by permutations among 3,4,5 and 1,2 separately. Other planar graphs are obtained from these by moving the nodes within the solid lines without disconnecting the diagram. For example 4 in the first diagram can be moved within the solid line 1-3.

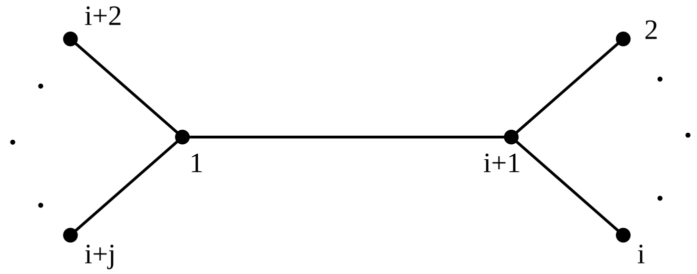

Figure 6: Simplest connected tree diagram for the loop counting purposes. The solid line between 1 and $i+1$ includes ϕ and ψ.

With a little more effort one can show that,

Theorem 1. *For scalar impurity BMN operators $\mathcal{O}(g^2_{\rm YM})$ interactions will not change the scaling law of (A.4) at all.*

Let us outline the proof shortly. As a first step, one can show that the only interactions involved in *scalar impurity* BMN operators are coming from F-terms in $\mathcal{N}=4$ SYM lagrangian. F and D type interaction terms written in $\mathcal{N}=1$ component notation reads,

$$F \propto (f^{abc} \bar{Z}^i_b Z^i_c)^2 \,, \qquad D \propto f^{abc} f^{ade} \epsilon_{ijk} \epsilon_{ilm} Z^j_b Z^k_c \bar{Z}^l_d \bar{Z}^m_e \,. \tag{A.5}$$

Here, $a,\dots$ are the color indices while $i,\dots$ denote flavor. Note that when one specifies the orientation in a scalar propagator $Z^i \bar{Z}^i$ as from Z to $\bar{Z}$, these quartic vertices can be

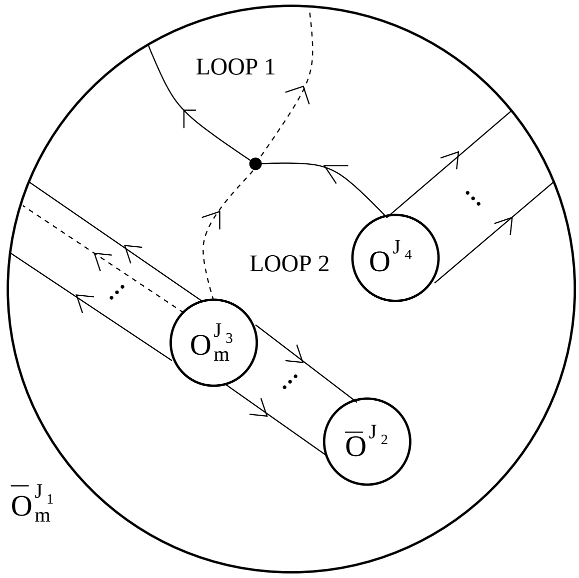

Figure 7: A quartic interaction in a typical diagram introduce two interaction loops. Here, loop 1 is contractible while loop 2 is non-contractible, therefore this diagram represents a semi-contractible interaction.

represented as in figure 8. In [17] it was shown that, correlation functions of BPS type multi-trace operators,

$$\mathrm{Tr}(Z_1^J)\cdots\mathrm{Tr}(Z_r^J) \tag{A.6}$$

do not receive any radiative corrections. To see this one first notes that F-type quartic vertex vanishes when fields are all have the same flavor. Secondly one discovers that contribution of D-type quartic vertex exactly cancels out the contributions from self energies and gluon exchange [17].

Now consider replacing some of the BPS operators in (A.6) with BMN operators, (1.2). Since the scalar impurities in BMN operators are distinguished from Z fields by their flavor, F-terms are now allowed. However, unlike F-type interactions D-term quartic vertex, gluon exchange and self energies are all flavor blind, therefore one can replace the ϕ and ψ impurities with Z fields for the sake of studying possible contributions from these interactions. After this replacement the phase sum over the position of impurities in O_n^J becomes trivial and factors out of the operator, hence BMN operators reduce to BPS type operators times an overall phase factor. Therefore the theorem of [17] for BMN type multi-trace operators becomes,

Theorem 2. *The only radiative corrections to 2-point functions of multi-trace BMN operators come from F-type interactions.*

Second step in the proof of theorem 1 is the classification of topologies of Feynman diagrams with one F-term interaction. Any $\mathcal{O}(\lambda')$ interaction that one inserts in (3.1) introduces two "interaction loops" in the plane diagrams. A generic example is shown in figure 7. According to the contractibility of these interaction loops one can classify planar F-term interactions as,

JHEP10(2003)027

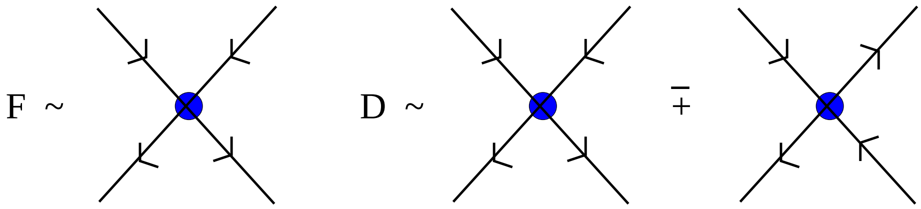

Figure 8: Orientations of F-term and D-term quartic vertices.

1. *Contractible*: both interaction loops are contractible,
2. *Semi-contractible*: only one of the loops is contractible,
3. *Non-contractible*: none of the loops are contractible.

Requirement of contractibility means that two incoming lines and two outgoing lines in the F-term vertex of figure 8 (i) belong to the same operator and (ii) adjacent to each other when drawn on a plane. Now, we note that this classification of interactions would hardly make any difference if we were not dealing with BMN type operators which involve a non-trivial phase summation over the position of the impurity. Structure of the F-term interactions in (A.5) makes it clear that interactions of adjacent lines yield a phase factor

$$1 - e^{\frac{2i\pi n}{J_1}} \sim \frac{1}{J} .$$

Therefore we learn that the phases in BMN operators provide a factor of $1/J^2$ for "contractible" interactions, $1/J$ for "semi-contractible" interactions, $1/J^0$ for "non-contractible" interactions.

As the last step in our proof of theorem 1, let us show that non-contractibility of each interaction loop supplies another factor of $1/J$. It should be clear from above requirements for contractibility that there are two distinct situations that non-contractibility of an interaction loop can arise:

1. the incoming (or outgoing) lines in figure 8 belong to different operators within a multi-trace operator *or*
2. belong to the same operator but are not adjacent to each other.

Let us now recall that among various contributions to the power of J in free correlators, there is a combinatorial factor of J^{i+j-3} coming from summing over positions of the insertions of O^{J_i} operators inside $\bar{O}^{J_i}$'s. In case 1 above, clearly, one of these position sums will be missing, hence a suppression by $1/J$. On the other hand case 2 can only arise in a situation where there is at least one operator inserted in between the incoming or (outgoing) lines which take place in the interaction. Since the position of this inserted operator is required to have a fixed position in between the interacting legs one also arrives at a suppression by $1/J$. These two situations are illustrated in an example of $G^{3,2}$ in figure 9.

When combined with the powers of J coming from the phase factors that we described above, we see that they compensate each other and one gets a universal factor $1/J^2$ for all of the different topologies in an F-term interaction, namely contractible, semi-contractible

JHEP10(2003)027

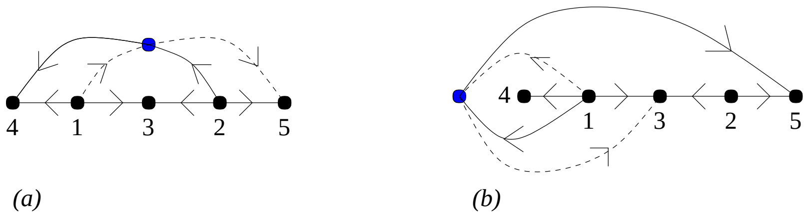

Figure 9: In (a) both of the interactions loops are non-contractible due to case 1: incoming and outgoing line pairs of the quartic vertex connect to different operators. In (b) one interaction loop is non-contractible due to case 1 the other due to case 2.

and non-contractible. Finally note that all interactions come with a factor of $g_{\rm YM}^2 N$. Combined with $1/J^2$ this yields λ' and therefore we concluded the proof of theorem 1: One-loop radiative corrections to (3.1) is of the form,

$$C^{i,j} \sim \frac{g_2^{i+j-2}}{J^{(i+j)/2-1}}\lambda' \,.$$

The reason that this theorem might fail in case of BMN operators with non-scalar impurities is that D-term interactions might give non-vanishing contributions (see [14]).

B. Field theory basis change at finite g_2

Apart from the purely field theoretical task of determining the eigenvalues and eigen-operators of $\Delta - J$, one can ask for the identification of the string basis in field theory for non-zero g_2 [9, 13]. For non-zero g_2, the inner product in field theory becomes non-diagonal in the original basis of BMN where there is an explicit identification of n-string states with n-trace operators. On the other hand, to identify the "string theory basis" in field theory one should require the field theory inner product to be diagonal for all g_2. However, this requirement alone *does not* uniquely specify the necessary basis change from BMN basis to string theory basis for finite g_2. One always has the freedom of performing an arbitrary unitary transformation.

In the recent literature, two independent but compatible approaches for the identification of string basis were presented. In the string bit formalism (SBF), it is possible to capture the kinematics and dynamics of gauge theory amplitudes by the discretized theory of bit strings.[16] According to the conjecture of [9], in the string bit language, the basis transformation which takes from BMN basis to string basis for all g_2 reads,

$$|\tilde{\psi}_i\rangle = \left(e^{-\frac{g_2}{2}\Sigma}\right)_{ij} |\psi_j\rangle \equiv (U_\Sigma)_{ij}\, |\psi_j\rangle \tag{B.1}$$

where ψ_n denotes an n-string state. Here Σ is the sum over all distinct transpositions of two string bits,

$$\Sigma = \frac{1}{J^2}\sum_{mn} \Sigma_{(mn)} \,.$$

Σ has the effect of a single string splitting or joining, i.e. it can map an i string state into an $i \pm 1$-string state. Note that the transformation matrix $e^{-\frac{g_2}{2}\Sigma}$ is real and symmetric.

[16] Strictly speaking this has been shown only at $\mathcal{O}(g_2^2\lambda')$ and only for scalar impurities [10].

Another method [13] which leads an identification of the string basis is simply to find the transformation U which diagonalizes the matrix of inner products between BMN operators, order by order in g_2. In the *free* theory, we define the following matrix,

$$G_{ij} = \langle \bar{O}_j O_j \rangle \,.$$

Here n is a collective index for a generic n-trace BMN operator. One identifies the basis transformation, U by requiring that G is diagonal in the new, "string basis":

$$U_{ik} G_{kl} U^\dagger_{lj} = \delta_{ij} \,, \qquad \tilde{O}_i \equiv U_{ij} O_j \,.$$

As in the SBF basis change U is specified up to an arbitrary unitary transformation. One can fix this freedom by requiring that U is real and symmetric. Call this transformation, U_G. Then the solution of the above equation up to $\mathcal{O}(g_2^2)$ reads,

$$U_G = \mathbf{1} + g_2 \left(\frac{1}{2} G^{(1)} \right) + g_2^2 \left(-\frac{1}{2} G^{(2)} + \frac{3}{8} (G^{(1)})^2 \right) + \cdots \tag{B.2}$$

where $G^{(i)}$ denotes $\mathcal{O}(g_2^i)$ piece of the metric.

The requirement of reality and symmetry completely fixes the freedom in the choice of the transformation. It was independently shown in [9] and [13] that this simple choice leads to an agreement with string theory calculations. In particular the inner product of a single and double trace operator in the *interacting* theory in this basis, agrees with the cubic string vertex. Recently, [12] also gives evidence for an agreement between the $\mathcal{O}(g_2^2 \lambda')$ eigenvalue of $\Delta - J$ and the matrix element of light cone hamiltonian in single string sector. Despite the agreement at this order, there is no reason to believe that this simple choice should hold at higher orders in g_2 *or* for higher trace multi-trace operators.

Here, we will confine ourselves to checking the compatibility of this basis transformation with free field theory correlators at g_2^2, including the effects of triple-trace operators. As a bonus we obtain new deconstruction identities decomposing some multi-trace inner products in terms of smaller multi-trace inner products. Since the requirement of reality and symmetry of the basis transformation matrix completely fixes the definition of the new basis, the two prescriptions described above should be equivalent. Equating various matrix elements of U_G and U_Σ we will obtain identities involving free field theory correlators which are subject to should be check in field theory.

The first non-trivial requirement is coming from the single-trace operators at $\mathcal{O}(g_2^2)$,

$$\langle \psi_1 | \tilde{\psi}_1 \rangle_G = \langle \psi_1 | \tilde{\psi}_1 \rangle_\Sigma \,.$$

L.h.s. is, (we suppress indices labeling a multi trace operator),

$$\langle \psi_1 | U_G | \psi_1 \rangle = 1 + g_2^2 \left(-\frac{1}{2} G^{11} + \frac{3}{8} G^{12} G^{21} \right) .$$

G^{11} is the torus level, free single-single correlator with the space-time dependence removed. Similarly G^{12} is $\mathcal{O}(g_2)$ level single-double correlator which is presented in eq. (2.9). R.h.s. reads,

$$\langle \psi_1 | 1 + \frac{1}{2} \left(-\frac{g_2^2}{2} \right)^2 \Sigma^2 | \psi_1 \rangle = \frac{g_2^2}{8} G^{12} G^{21} \,,$$

where we used the fact that Σ changes the string number by ± 1. Hence we obtain,

$$G^{11}_{nm} = \frac{1}{2}\sum_i G^{12}_{n;i} G^{12}_{i;m}\,. \tag{B.3}$$

Here i is a collective index labeling either of the two types of double-trace operators which can appear in the intermediate process i.e. $i = \{ny\}$ for BMN double-trace and $i = y$ for BPS double-trace.

This identity indeed holds in the gauge theory as first shown in [15]. With the inverse reasoning we can view this simple calculation as the *derivation* of this non-trivial sum formula. As a next application let us derive identities involving triple trace operators. We require,

$$\langle\psi_3|\tilde{\psi_1}\rangle_G = \langle\psi_3|\tilde{\psi_1}\rangle_\Sigma\,.$$

Much as above, $O(g_2^2)$ term in l.h.s. is,

$$-\frac{1}{2}G^{13} + \frac{3}{8}G^{12}G^{23}$$

and the r.h.s. is,

$$\frac{g_2^2}{8}\langle\psi_3|\Sigma^2|\psi_1\rangle = \frac{g_2^2}{8}G^{12}G^{23}\,.$$

We learn that

$$G^{13} = \frac{1}{2}G^{12}G^{23}\,.$$

Notice that "2" in the intermediate step can not be a BPS double-trace operator, because the lowest order G^{23} between a BPS double-trace and a BMN triple trace is at $\mathcal{O}(g_2^3)$.[17] Therefore we arrive at the formula,

$$G^{13}_{m;nyz} = \frac{1}{2}\sum_{py'} G^{12}_{m;py'} G^{23}_{py';nyz}\,. \tag{B.4}$$

The single-triple correlation function G^{13} is computed in appendix D. Again this identity is subject to check by a direct field theory computation. This is done in appendix C and (B.4) passes the test. A similar calculation with the requirement $\langle\psi_2|\tilde{\psi_2}\rangle_G = \langle\psi_2|\tilde{\psi_2}\rangle_\Sigma$, one reaches at another useful identity,

$$G^{22}_{my;m'y'} = \frac{1}{2}\left(\sum_n G^{12}_{my;n}G^{12}_{n;m'y'} + \sum_i G^{23}_{my;i}G^{23}_{i;m'y'}\right)\,. \tag{B.5}$$

Here $i = \{ny''z''\}$ or $i = \{y''z''\}$ for BMN and BPS triple-trace operators. An expression for the free $\mathcal{O}(g_2^2)$ double-double correlator, G^{22}, is given in eq. (4.13) We also checked this by direct computation in appendix C. We emphasize that these are highly non-trivial identities

[17] It is easy to see (either by using trace algebra or by counting the loops in Feynman diagrams) that there exists only *disconnected* $\mathcal{O}(g_2)$ contributions to any double-triple correlator where the 2-3 correlator separates as 1-2, and 1-1. This obviously cannot happen for a correlation function of a BPS double-trace operator and a BMN triple-trace operator.

JHEP10(2003)027

viewed as representation of a trigonometric function, say G^{13} in (2.12) as an infinite series of products of simpler functions. In comparison to (B.3) the non-triviality comes from the fact that summands are trigonometric functions rather that rational functions as in the r.h.s. of (B.3). These identities will prove extremely useful for the computations of the next sections. Finally we note immediate generalizations,

$$\begin{aligned} G^{m,m+2} &= \frac{1}{2} G^{m,m+1} G^{m+1,m+2} \\ G^{mm} &= \frac{1}{2}\left(G^{m,m-1}G^{m-1,m} + G^{m,m+1}G^{m+1,m}\right). \end{aligned}$$

C. Sum formulas

Let us first reproduce the matrices $A_{mm'}$ and $B_{mm'}$ that appear in the $\mathcal{O}(g_2^2)$ and $\mathcal{O}(g_2^2\lambda')$ pieces of the single-trace two-point function. These were defined in [4]:

$$A_{m,n} = \begin{cases} \frac{1}{24}, & m = n = 0\,; \\ 0, & m = 0, n \neq 0 \text{ or, } \; n = 0, \; m \neq 0\,; \\ \frac{1}{60} - \frac{1}{6u^2} + \frac{7}{u^4}, & m = n \neq 0\,; \\ \frac{1}{4u^2}\left(\frac{1}{3} + \frac{35}{2u^2}\right), & m = -n \neq 0\,; \\ \frac{1}{(u-v)^2}\left(\frac{1}{3} + \frac{4}{v^2} + \frac{4}{u^2} - \frac{6}{uv} - \frac{2}{(u-v)^2}\right), & \text{all other cases} \end{cases} \tag{C.1}$$

and

$$4\pi^2 B_{m,n} = \begin{cases} 0\,, & m = n = 0\,; \\ \frac{1}{3} + \frac{10}{u^2}\,, & m = n \neq 0\,; \\ -\frac{15}{2u^2}\,, & m = -n \neq 0\,; \\ \frac{6}{uv} + \frac{2}{(u-v)^2}\,, & \text{all other cases} \end{cases} \tag{C.2}$$

where $u = 2\pi m, u = 2\pi n$.

The rest of this appendix outlines the computation of non-trivial summations that appear along the computations in sections 2, 4, 6 and the previous appendix. We will first prove that (B.4) and (B.5) indeed hold in GT. These were obtained in the previous appendix simply by equating the basis transformations U_Σ and U_G. A second task is to derive (2.21) of section 4. Finally we will define and evaluate the last term in (5.3) of section 5.

We reproduce (B.4) here for completeness:

$$G^{13}_{m;nyz} = \frac{1}{2}\sum_{py'} G^{12}_{m;py'} G^{23}_{py';nyz}\,. \tag{C.3}$$

L.h.s. of (C.3) is computed in appendix D and presented in eq. (2.11). G^{23} that appear on the r.h.s. gets $\mathcal{O}(g_2)$ contributions only from disconnected diagrams as $2 \to 3$ decouples as, $1 \to 2$ and $1 \to 1$. This quantity is also computed in appendix D, result is given in (2.14). One gets two contributions to r.h.s. of (C.3) from first and second pieces in (2.14). Second contribution gives,

$$\frac{1}{2}\int_0^1 \mathrm{d}y' \sum_{p=-\infty}^{\infty} G^{12}_{m,py'} \frac{g_2}{\sqrt{J}} \delta_{np}\delta_{yy'} \sqrt{(1-y)z\tilde{z}} = \frac{1}{2}\frac{g_2}{\sqrt{J}}\sqrt{(1-y)z\tilde{z}}\, G^{12}_{m,ny}\,. \tag{C.4}$$

JHEP10(2003)027

First piece in G^{23} gives,

$$\frac{1}{2}\sqrt{\frac{z\tilde{z}}{y}}\int_0^1 \mathrm{d}y' y'^3(\delta_{y',y+z}+\delta_{y',y+\tilde{z}})\frac{\sin^2(m\pi y')}{\pi^4}\left\{\sum_{p=-\infty}^{\infty}\frac{sin^2(p\pi y/y')}{(p-my')^2(p-ny'/y)^2}\right\}. \quad \text{(C.5)}$$

We will now describe the evaluation of the sum in this expression. Let us separate the sum into two pieces as,

$$S = S_1 + S_2 \equiv \sum_{p=-\infty}^{\infty}\frac{1}{(p-a)^2(p-b)^2} - \sum_{p=-\infty}^{\infty}\frac{\cos^2(px)}{(p-a)^2(p-b)^2}\,, \quad \text{(C.6)}$$

where we defined,

$$x \equiv \frac{\pi y}{y'}\,, \qquad a \equiv my'\,, \qquad b \equiv \frac{ny'}{y}\,.$$

S_1 is easy to evaluate (can be done with a computer code) and the result is,

$$S_1 = \frac{1}{(a-b)^3}\left(2\pi\cot(\pi a) - 2\pi\cot(\pi b) + \frac{(a-b)\pi^2}{\sin^2(\pi a)} + \frac{(a-b)\pi^2}{\sin^2(\pi b)}\right). \quad \text{(C.7)}$$

It is not possible to evaluate S_2 neither with a well-known computer program nor it can be found in standard tables of infinite series (like Gradhsteyn-Rhyzik or Prudnikov). To tackle with it we reduce it into a product of two sums as,

$$\begin{aligned} S_2 &= \frac{d^2}{dadb}\left(\sum_{p=-\infty}^{\infty}\frac{\cos(px)}{(p-a)}\right)\left(\sum_{r=-\infty}^{\infty}\frac{\cos(rx)}{(r-b)}\delta_{pr}\right) \\ &= \frac{d^2}{dadb}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\left(\sum_{p=-\infty}^{\infty}\frac{\cos(px)e^{ipt}}{(p-a)}\right)\left(\sum_{r=-\infty}^{\infty}\frac{\cos(rx)e^{-irt}}{(r-b)}\right), \end{aligned}$$

where in the second step we used the integral representation of δ_{pr}. Expanding the exponentials in terms of cos and sin we now reduced the sum into sums of the following form,

$$\sum_{p=-\infty}^{\infty}\frac{\cos(p(x\pm t))}{p-a} = -\frac{1}{a} + 2a\sum_{p=1}^{\infty}\frac{\cos(p(x\pm t))}{p^2-a^2} = -\frac{1}{a} + 2af_m(x\pm t, a)\,. \quad \text{(C.8)}$$

We can read off the function $f_m(z,a)$ from e.g. [18],

$$f_m(z,a) = \frac{1}{2a^2} - \frac{\pi}{2a}\frac{\cos(a(z-(2m+1)\pi))}{\sin(\pi a)}\,, \quad \text{(C.9)}$$

where m is an integer defined as, $2\pi m \le z \le 2\pi(m+1)$. With the given information it is straightforward to evaluate these sums. Integrating over t and combining with S_1 in (C.7), one gets,

$$\begin{aligned} \sin^2(\pi a)S &= \frac{\pi}{2(a-b)^3}\left[\sin(2\pi a) - \sin(2ax) - \sin(2a(\pi-x))\right] + \\ &\quad + \frac{\pi}{(a-b)^2}\left[x\sin^2(\pi a) + x\sin^2(a(\pi-x)) + (\pi-x)\sin^2(ax)\right]. \end{aligned}$$

We insert this expression into (C.5), carry out the trivial integration over y'. Using the definitions of x, a and b given above one gets,

$$-\frac{\pi}{2(m-k)^3}\left\{\sin(2\pi mz)+\sin(2\pi m\tilde{z})+\sin(2\pi my)\right\}+$$
$$+\frac{\pi^2}{(m-k)^2}y(\sin^2(\pi mz)+\sin^2(\pi m\tilde{z}))+\frac{\pi^2}{2(m-k)^2}(1-y)\sin^2(\pi my)\,,$$

where $k=n/y$. Comparison of this expression with (2.11) shows that this expression equals,

$$G^{13}_{m;nyz}-\frac{g_2}{2}\sqrt{\frac{z\tilde{z}(1-y)}{J}}G^{12}_{m,ny}\,.$$

Adding up to this the first contribution in (C.4) we proved (C.3).

Now, let us move on the proof of the second decomposition identity, (B.5) that we reproduce here,

$$G^{22}_{m,y;m',y'}=\frac{1}{2}\left(\sum_n G^{12}_{m,y;n}G^{12}_{n;m'y'}+\sum_i G^{23}_{m,y;i}G^{23}_{i;m'y'}\right). \tag{C.10}$$

As mentioned before, there are disconnected and connected contributions to both l.h.s. and r.h.s. of this equation. Since connected contributions differ from the disconnected ones by a factor of $1/J$, one should match $\mathcal{O}(1)$ and $\mathcal{O}(1/J)$ pieces on both sides separately. Here we will present the equality of $\mathcal{O}(1)$ parts of l.h.s. and r.h.s. and leave the question of $\mathcal{O}(1/J)$ pieces for future. We did not need $\mathcal{O}(1/J)$ terms anywhere in our computations. $\mathcal{O}(g_2^2)$ disconnected contribution to G^{22} is just

$$\langle\bar{O}^{J_1}_n O^{J_3}_m\rangle_{g_2^2}\langle\bar{O}^{J_2}O^{J_4}\rangle_{g_2^0}$$

plus

$$\langle\bar{O}^{J_1}_n O^{J_3}_m\rangle_{g_2^0}\langle\bar{O}^{J_2}O^{J_4}\rangle_{g_2^2}\,.$$

All required terms here were already computed in the literature (see [2, 4]) and the total result is,

$$g_2^2\left(y^4A_{mm'}+\frac{\delta_{mm'}}{24}(1-y)^4\right).$$

Turning to the r.h.s. of (C.10) now, we first note that $2\to1\to2$ process can not be disconnected hence does not contribute at this order. Evaluation of the second term in r.h.s. is straightforward by using (2.14) that is derived in the next appendix. The triple-race that appears in the intermediate step can either be a BMN or a BPS operator. Let us first consider the former case. We need to compute,

$$\begin{aligned}G^{23}_{my;pst}G^{23}_{pst,m'y'}\ =&\ \left(y^{3/2}G^{12}_{m;ps/y}(\delta_{y,s+t}+\delta_{y,1-t})+\frac{g_2}{\sqrt{J}}\delta_{mp}\delta_{ys}\sqrt{(1-y)t\tilde{t}}\right)\times\\ &\times\left(y'^{3/2}G^{12}_{m';ps/y'}(\delta_{y',s+t}+\delta_{y',1-t})+\frac{g_2}{\sqrt{J}}\delta_{m'p}\delta_{y's}\sqrt{(1-y')t\tilde{t}}\right),\end{aligned}$$

where one sums over pst. When last term in the first parenthesis goes with the last term of the second we have the expression,

$$\frac{g_2^2}{J}\sum_{p=-\infty}^{\infty}\delta_{pm}\delta_{pm'}\left(J\int_0^1 \mathrm{d}s\right)\left(\frac{J}{2}\int_0^{1-y}\mathrm{d}t\right)\times$$
$$\times\delta_{sy}\delta_{sy'}\sqrt{(1-y)(1-y')}t(1-y-t) = \delta_{yy'}\delta_{mm'}\frac{(1-y)^4}{24}, \qquad \text{(C.11)}$$

where in the integral over t we divided by a factor of 2 to reconcile with the double-counting (note that $t\to 1-t$ is not distinguishable at the level of Feynman diagrams when triple trace is BPS, and one should divide out the symmetry factor). When first term in the first parenthesis goes with second or third terms of the second, both of the integrals over s and t are constraint by the delta-functions and one gets a $1/J$ suppression. A similar remark apply the case when second goes with third. Therefore we see that all cross terms are suppressed and only other non-vanishing contribution comes by matching second with second and third with third. This is,

$$\frac{1}{2}2(yy')^{3/2}\delta_{yy'}\sum_{p=-\infty}^{\infty}\int_0^y \mathrm{d}sG^{12}_{m,ps/y}G^{12}_{m',ps/y'} = y^4\sum_{p=-\infty}^{\infty}\int_0^1 \mathrm{d}xG^{12}_{m,px}G^{12}_{m',px}$$

where we again divided out a similar symmetry factor. It is easy to see that when the intermediate triple-trace operators are BPS type one gets the expression,

$$y^4\sum_{p=-\infty}^{\infty}\int_0^1 \mathrm{d}xG^{12}_{m,x}G^{12}_{m',x}$$

instead of the above expression. Adding these two up and using (B.3), one gets $2y^4G^{11}_{mm'}$. Combining it with the contribution from (C.11) and comparing with (4.13) for the case of $i=2$ we proved (B.5) at the leading order.

Next, we shall present two new "interacting level" decomposition identities which are essentially the analogs of the identities given in appendix D of [8]:

$$\sum_{p,y'}\frac{p}{y'}G_{n,py'}G_{py',myz} = \left(n+\frac{m}{y}\right)G_{n,myz}, \qquad \text{(C.12)}$$

$$\sum_{p,y'}\frac{p^2}{y'^2}G_{n,py'}G_{py',myz} = \left(n^2+\left(\frac{m}{y}\right)^2\right)G_{n,myz}+B^{13}_{n,myz} \qquad \text{(C.13)}$$

where B^{13} is given in (2.13). These identities can easily be proven by the methods described above.

As an application of these decomposition identities let us prove (2.17). For notational simplicity we will no show the indices fully in the following e.g. we denote Γ_n^{myz} as Γ_1^3, etc. Eq. (2.18) gives,

$$\begin{aligned}\Gamma_1^3 &= G^{33}\Gamma_{31}+G^{32}\Gamma_{21}+G^{31}\Gamma_{11}\\ &= \frac{1}{2}\frac{m}{y}\left(\frac{m}{y}-n\right)G^{13}+\frac{1}{2}B^{13}-\frac{1}{2}\sum_{p,y'}\frac{p}{y'}\left(\frac{p}{y'}-n\right)G^{12}G^{23}.\end{aligned}$$

where the second line follows after trivial algebra. Now, using (B.4), (C.12) and (C.13) it is immediate to see that

$$\Gamma_1^3 = 0\,.$$

Let us now describe the evaluation of (2.21) in section 2. We separate the l.h.s. of (2.21) into two parts as,

$$\sum_p(-nA(p)+B(p)) \equiv \sum_p \left(-n\frac{\sin^2(\pi py/y')}{(n-p/y')^2(n^2-(p/y')^2)} + \frac{\sin^2(\pi py/y')}{(n-p/y')^3}\right). \tag{C.14}$$

Evaluation of $\sum_p B(p)$ is easier. It can be written as,

$$\sum_p B(p) = \frac{y'^3}{2}\left(-\sum_{p=-\infty}^{\infty}\frac{1}{(p-a)^3} + \frac{1}{2}\frac{d^2}{da^2}\left\{2a\sum_{p=1}^{\infty}\frac{\cos(px)}{p^2-a^2} - \frac{1}{a}\right\}\right).$$

Each of the sums can be found in standard tables such as [18] and the result is,

$$\sum_p B(p) = \pi^3 y^2 y' \cot(\pi n y')\,. \tag{C.15}$$

To compute $\sum_p A(p)$ we write it as,

$$A(p) = \frac{y'^4}{2}\left(\frac{\cos(px)}{(p-a)^2(p^2-a^2)} - \frac{1}{(p-a)^2(p^2-a^2)}\right).$$

Second can be done by a standard computer code. First can be written as,

$$\sum_p \frac{\cos(px)}{(p-a)^2(p^2-a^2)} = -\frac{1}{a^4} + \left(\frac{d^2}{dx^2}+a^2\right)\frac{d^2}{db^2}\sum_{p=1}^{\infty}\frac{\cos(px)}{(p^2-b)}$$

where $b = \sqrt{a}$. This can be looked up in [18]. Combining the result with (C.15) one obtains (2.21).

D. Computation of G^{13}, G^{23}, Γ^{13} and Γ^{23}

We will first describe the evaluation of free planar single-triple correlator at the planar level, $\langle \bar{O}_n^J : O_m^{J_1} O^{J_2} O^{J_3} :\rangle$. We will refer to the operators that appear in this expression as “big operator”, “operator 1”, “operator 2” and “operator 3”, respectively. Let us also denote the ratios of the “sizes” of these operators by

$$y = \frac{J_1}{J}\,, \qquad z = \frac{J_2}{J}\,, \qquad \tilde{z} = \frac{J_3}{J}\,. \tag{D.1}$$

Since the space-time dependence of two-point functions of scalar operators is trivial we will only be interested in the coefficient that multiplies the space-time factors, i.e. G^{13} and Γ^{13}. Nevertheless, let us show the space-time factors here, for completeness. For the free case it is just product of $J+2$ scalar propagators,

$$\frac{1}{(4\pi^2x^2)^{J+2}}\,.$$

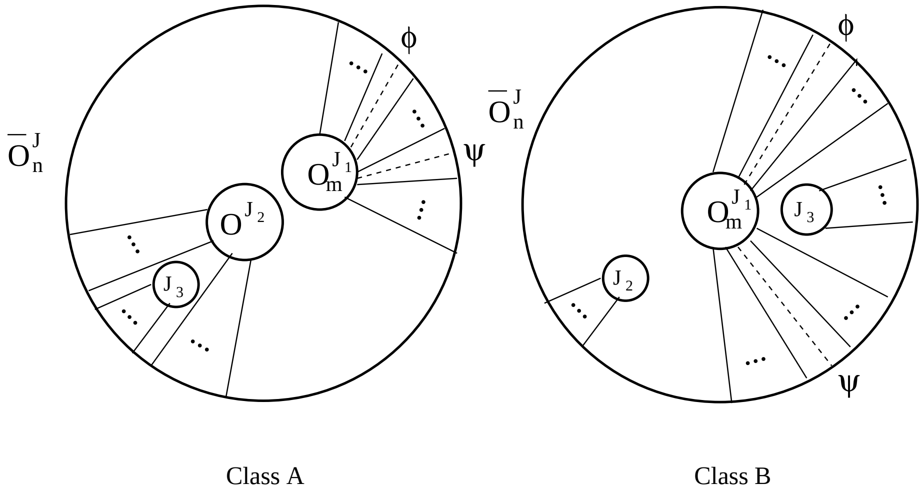

Figure 10: Two different classes of free diagrams. Dashed lines denote propagators of impurity fields.

In case of one-loop interactions, one needs to perform the following interaction over the position of the vertex,

$$\frac{1}{16\pi^4}\int\frac{\mathrm{d}^4y}{y^4(y-x)^4}=\frac{\ln(\Lambda^2x^2)}{8\pi^2x^4}\,.$$

Therefore the space-time dependence at $\mathcal{O}(\lambda')$ is,

$$\frac{1}{8\pi^2(4\pi^2x^2)^{J+2}}\ln(\Lambda^2x^2)\,.$$

Let us now describe the evaluation of the coefficients that multiply these space-time factors.

General strategy is first to fix the position of one operator, say 2 inside the big operator. Then we are left with phase sums over positions of both of the impurities and the position of operator 3 inside the big operator. Of course one still has to take into account the cyclicity of 2 and 3 which yield a multiplicative factor of J_2J_3. After fixing the position of 2, we can divide the planar diagrams into two classes.

In *class A* (see figure 10.) operator 2 is “outside” the bunch of lines connecting operator 1 to operator the big operator, hence the phase sum over ϕ and ψ is trivial:

$$J_1^2\int_0^1\mathrm{d}a\int_0^1\mathrm{d}b e^{2i\pi(m-ny)a}e^{-2i\pi(m-ny)b}=\frac{\sin^2(2\pi ny)}{\pi^2(ny-m)^2}.$$

One also has to sum over the position of operator 3 “under” 2 and position of operator 2 “under” 3. This gives a combinatorial factor of J_2+J_3. As apparent from figure 10. class A is equivalent to single-double correlator up to the aforementioned overall combinatorial factor. Therefore the left diagram in figure 10. equals,

$$G_A^{13}=\frac{g_2^2}{J}(1-y)\sqrt{\frac{z\tilde{z}}{y}}\frac{\sin^2(2\pi ny)}{\pi^2(n-k)^2}\qquad\text{(D.2)}$$

where we took into account the normalization of operators and defined $k\equiv m/y$.

In *class B* (see figure 10), operator 3 is inserted inside the bunch of lines connecting 1 to the big operator, hence the phase summations become non-trivial. One fixes the position of 2 inside the bunch, then sums over the positions of ϕ and ψ. As one impurity jumps over operator 3 one gets an enhancement in the phase of the big operator by a factor of $-2\pi i n\tilde{z}$. One evaluates the sums taking this point into account, than one sums over the position of operator 2. This procedure gives,

$$\begin{aligned} G_B^{13} &= \frac{g_2^2}{\pi^2 J}\sqrt{\frac{z\tilde{z}}{y}}\frac{1}{(n-k)^2}\Bigg(y(\sin^2(\pi n z)+\sin^2(\pi n\tilde{z}))- \\ &\qquad -\frac{1}{2\pi(n-k)}(\sin(2\pi n y)+\sin(2\pi n z)+\sin(2\pi n\tilde{z}))\Bigg). \end{aligned} \tag{D.3}$$

Adding up (D.2) and (D.3) gives eq. (2.11)[18]

There are two consistency checks that one can perform. First of all — as apparent from the diagrams — the final expression should be symmetric in $J_2 \leftrightarrow J_3$. (2.11) nicely passes this test. A more non-trivial test is to check whether G^{13} reduces to G^{12} as one takes $J_3 \to 0$. Straightforward algebra shows that,

$$G^{13}_{n,myz} \to \sqrt{\frac{z}{J}}G^{12}_{n,my}$$

and confirms our expectation.

Now let us discuss how to add interactions to figures10, by preserving *planarity.* As already mentioned for the evaluation of general correlators in appendix A, there are three distinct classes of planar interactions: *contractible, semi-contractible and non-contractible.* Above we noted that evaluation of class A diagrams are completely equivalent to single-double correlator. This continues to be the case when one introduces planar interactions. There are only contractible and semi-contractible contributions in this case and since $\mathcal{O}(\lambda')$ corrections to this correlator was already computed in [8] (that we reproduced in eq. (2.10)), we will only show the final result,

$$\text{Class A interactions} \;\Rightarrow \frac{g_2^2\lambda'}{J\pi^2}(1-y)\sqrt{\frac{z\tilde{z}}{y}}\frac{\sin^2(2\pi n y)}{\pi^2(n-k)^2}(k^2-nk+n^2)\,. \tag{D.4}$$

Let us explain the evaluation of interactions in class B in some detail. Contractible interactions are coming from the situation where an impurity interacts with its nearest-neighbor in such a way that both interaction loops are contractible. As described in appendix A this gives a phase factor of

$$\left(1-e^{\frac{-2i\pi n}{J}}\right)\left(1-e^{\frac{2i\pi m}{J_1}}\right)\approx\frac{4\pi^2}{J^2}nk \tag{D.5}$$

for each possible nearest-neighbour interaction. One should sum over the insertions of this interaction between all adjacent line pairs between operator 1 and the big operator in

[18] We thank Neil Constable who computed this quantity by a completely different method (direct evaluation of the traces over the color structure and extracting out the $\mathcal{O}(g_2^2)$ piece) and who obtained the same result.

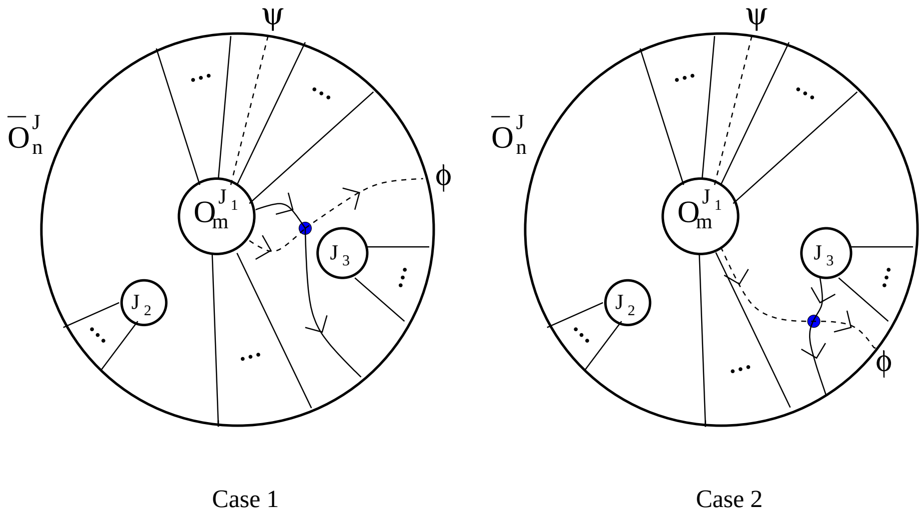

Figure 11: Two different types of semi-contractible diagrams. One should also consider the cases when two incoming and outgoing lines are interchanged. Also analogous contributions come from exchanging operator 2 with operator 3.

figure 10b, *except* the particular position when this line pair coincides with the position of operator 3. In this particular case one gets a semi-contractible diagram (see figure 11a). This sum procedure obviously gives the phase factor in (D.5) times (D.3).

Semi-nearest interactions in class B arise in two possible ways. First possibility is already mentioned above and shown in figure 11a. Another possibility arise when one of the interaction loops is non-contractible for another reason: the line pair that is incoming to the vertex connect to different operators. This is illustrated in figure 11b. Evaluation of the phase factors in both of these cases is simple:

In case 1, when one takes all possible orderings of the interacting pairs of lines, one gets a factor of

$$\left(1 - e^{2i\pi n\tilde{z}}\right)\left(1 - e^{-\frac{2i\pi m}{J_1}}\right)$$

in place of (D.5). Next, one has to sum over all possible positions of operator 3 in figure 11a. Finally one evaluates the phase sum over ψ. There is an analogous contribution where ψ impurity takes place in the interaction instead of ϕ. But that is obviously obtained from the former just by taking the complex conjugate.

In case 2, figure 11, one has to sum over the possibilities where ϕ interacts with the leftmost and the rightmost Z line in operator 3. Considering also the two different orderings of the ϕ and Z that are outgoing from the vertex, one obtains an overall phase factor of,

$$\left(1 - e^{\frac{-2i\pi n}{J}}\right)\left(1 - e^{2i\pi n\tilde{z}}\right).$$

Next, just as in the case 1 above, one sums over all possible insertions of operator 3 and positions of ψ impurity. Similarly one considers the conjugate case where ψ takes place in the interaction instead of ϕ. Finally one gets similar expressions to the ones obtained in case 1 and case 2 by exchanging the roles played by operator 3 and operator 2.

Combining all of the results above, namely both contractible and semi-contractible contributions in class A given in (D.4), contractible contributions in class B and all semi-nearest contributions in class B, one obtains a surprisingly simple expression. All of the factors conspire to give,

$$\Gamma_A^{13} + \Gamma_{B,\text{cont.}}^{13} + \Gamma_{B,\text{semi-cont.}}^{13} = \lambda'(n^2 - nk + k^2)G^{13}\,. \tag{D.6}$$

A few observations are in order. Notice that one obtains the same form for the interacting single-double correlator as shown in [8], see eq. (2.10). The proportionality to G^{12} (or G^{13} in our case) is obvious from the beginning. Because eventually, the effect of interactions is to dress the free expression with an overall phase factor. The surprise is that this phase factor,

$$n^2 - nk + k^2\,,$$

is the same in the cases of single-double and single-triple correlators! One appreciates the non-triviality of this after seeing the delicate conspiring of many different terms in our case. We also see the same phase dependence, in case of double-triple correlator, eq. (2.16). It is tempting to believe that this remains to be true in case of general i-trace j-trace correlator. Namely we believe that the result of contractible and semi-contractible interactions at $\mathcal{O}(\lambda')$ for more general extremal correlators can be summarized as,

$$\Gamma^{i,j}_{\text{cont. + semi-cont.}} = \lambda'(n^2 - nk + k^2)G^{i,j}\,.$$

Actually it suffices to see this behaviour in case of extremal correlators of the type $\Gamma^{1,i}$ since $\Gamma^{i,j}$ can be related to this by the disconnectedness argument.

However, this is not the whole story. There is a very important new class of planar diagrams which contributes to G^{13}: *non-contractible* diagrams. This was absent in the case of Γ^{12} because there was not enough number of operators to create this new interaction topology. This will become clear in the following.

We show all possible non-contractible diagrams in figure 12. Note that both of the interaction loops are non-contractible in this case. The loop formed by incoming lines is non-contractible because they belong to different operators. The loop formed by the outgoing lines is non-contractible because there is an operator inserted between them. This exemplifies our schematic discussion about the non-contractibility of planar diagrams in appendix A where we referred to these possibilities as case 1 and 2. At first sight one expects that these diagrams be suppressed by a factor of $1/J^2$ when compared with the contractible diagrams or by a factor of $1/J$ when compared with the semi-contractible diagrams, because the sums over the position of the operator 3 and the position of ϕ impurity are missing. However as noted in the general discussion of appendix A, there is a compensating enhancement coming from the overall phase factors associated with these diagrams, namely the $O(1/J^2)$ phase suppression given by (D.5) is absent. Therefore these diagrams are on the equal footing with the rest i.e. (D.6).

The evaluation of non-contractible diagrams is the simplest. One adds up all possible contributions that are displayed in figure 12, and include the analogous cases where one

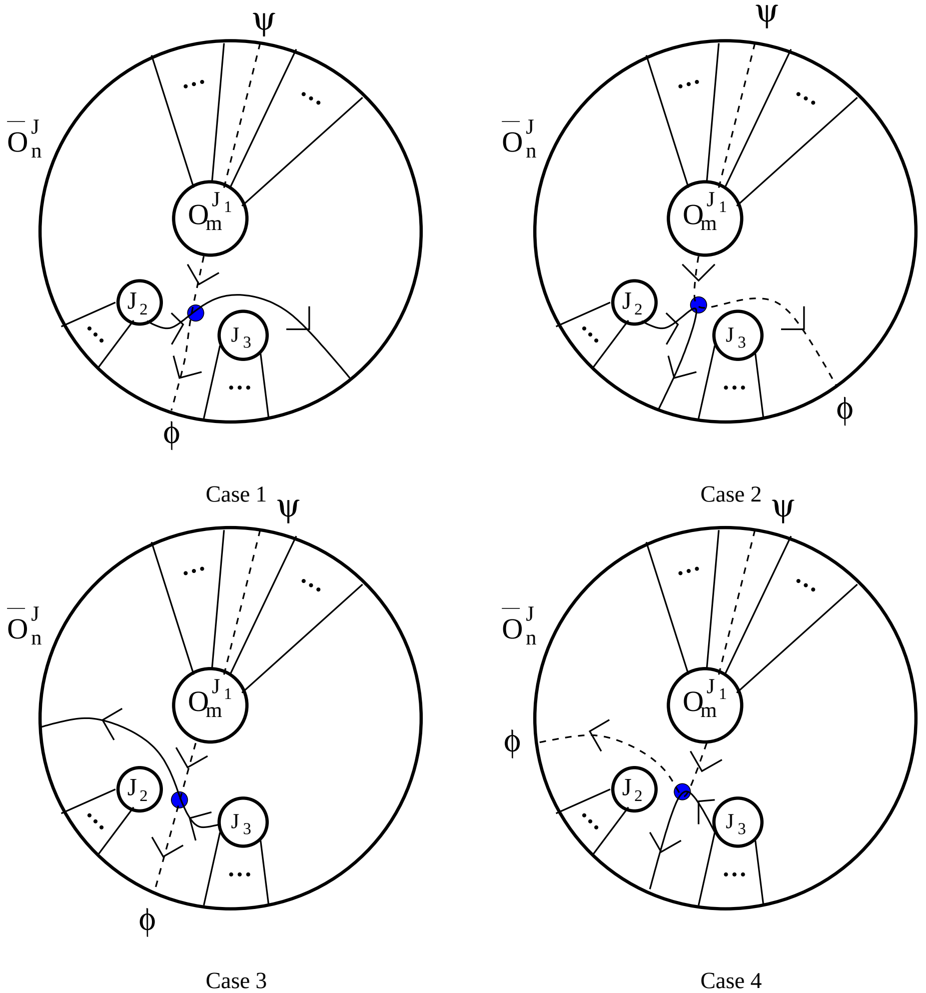

Figure 12: Contributions form non-contractible planar interactions. There are four more diagrams which are obtained by exchanging operator 2 with operator 3.

interchanges operator 2 with operator 3. Finally one performs the phase summation over ψ. Adding this result with the conjugate one which is obtained by interchanging the roles of ϕ and ψ, one gets (2.13).

Adding up (2.13) with (D.6), one obtains the total $\mathcal{O}(g_2^2\lambda')$ single-triple correlator, (2.12). Again, there are two consistency checks that one can perform. Firstly it is easy to see that, (2.12) is symmetric in $J_2 \leftrightarrow J_3$. Secondly when one takes the limit $J_3 \to 0$, B^{13} vanishes (as it should) and the rest of the expression boils down to the single-double result,

$$\Gamma^{13} \to \sqrt{\frac{z}{J}}\Gamma^{12}\,.$$

Let us now explain the computations that lead to the expressions in (2.14) and (2.16). When compared with the evaluation of single-triple correlators, evaluation of lowest order G^{23} and Γ^{23} is almost trivial. This is because only the disconnected diagrams contribute to these correlators at $\mathcal{O}(g_2)$. We now describe the evaluation of G^{23}. It will suffice to describe possible ways that $2 \to 3$ correlator can be separated into $1 \to 1$ and $1 \to 2$. Consider the correlator $\langle : \bar{O}_n^{J_1} \bar{O}^{J_2} :: O_m^{J_3} O^{J_4} O^{J_5} : \rangle$ define the ratios of lengths of the operators,

$$y = \frac{J_1}{J}\,, \qquad y' = \frac{J_3}{J}\,, \qquad z' = \frac{J_4}{J}\,, \qquad \tilde{z}' = \frac{J_5}{J}\,. \tag{D.7}$$

Since the impurity fields in operator 1 and operator 3 should be Wick contracted with each other, the only disconnected contributions arise when,

1. 1 connects to 3, 2 connects to 4 and 5,
2. 1 connects to 3 and 4, 2 connects to 5,
3. 1 connects to 3 and 5, 2 connects to 4.

A simple loop counting shows that all other contractions will result in higher orders in g_2. Cases 2 and 3 are easily expressible in terms of the results already reported in the literature, (see [4]). Therefore we only show case 1 which turns out to be the simplest,

$$\langle : \bar{O}_n^{J_1} \bar{O}^{J_2} :: O_m^{J_3} O^{J_4} O^{J_5} : \rangle_1 = \langle \bar{O}_n^{J_1} O_m^{J_3} \rangle \langle \bar{O}^{J_2} : O^{J_4} O^{J_5} : \rangle\,,$$

where one needs the lowest order contributions to the correlators on r.h.s. First one is just δ_{nm}. One evaluates the BPS correlator above by noting that the cyclicity factor of $J_2 J_4 J_5$ and the normalizations; hence one gets,

$$\frac{g_2}{\sqrt{J}} \sqrt{(1-y) z' \tilde{z}'}\,.$$

Combining this contribution with cases 2 and 3, one easily obtains, (2.14).

Computation of Γ^{23} at the lowest order, $\mathcal{O}(g_2 \lambda')$, goes by inserting planar interactions into the cases 1,2, and 3 that we listed above in all possible ways. In case 1, interactions can only be inserted in the correlator 1-3 since 2-4+5 — being a BPS corrrelator — does not receive radiative corrections. For the same reason interactions can be inserted only in the first correlators in cases 2 and 3. Necessary computations were already done in the literature (see e.g. [4, 8]) and one immediately gets, (2.16).

References

[1] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[2] C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and PP-wave strings*, *Nucl. Phys.* **B 643** (2002) 3 [hep-th/0205033].

[3] D.J. Gross, A. Mikhailov and R. Roiban, *Operators with large R charge in $N = 4$ Yang-Mills theory*, *Ann. Phys. (NY)* **301** (2002) 31 [hep-th/0205066].

[4] N.R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, *PP-wave string interactions from perturbative Yang-Mills theory*, *J. High Energy Phys.* **07** (2002) 017 [hep-th/0205089].

[5] A. Santambrogio and D. Zanon, *Exact anomalous dimensions of $N = 4$ Yang-Mills operators with large R charge*, *Phys. Lett.* **B 545** (2002) 425 [hep-th/0206079].

[6] D.J. Gross, A. Mikhailov and R. Roiban, *A calculation of the plane wave string hamiltonian from $N = 4$ super-Yang-Mills theory*, *J. High Energy Phys.* **05** (2003) 025 [hep-th/0208231].

[7] N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *BMN correlators and operator mixing in $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 650** (2003) 125 [hep-th/0208178].

[8] N.R. Constable, D.Z. Freedman, M. Headrick and S. Minwalla, *Operator mixing and the BMN correspondence*, *J. High Energy Phys.* **10** (2002) 068 [hep-th/0209002].

[9] J. Pearson, M. Spradlin, D. Vaman, H. Verlinde and A. Volovich, *Tracing the string: bmn correspondence at finite J^2/N*, *J. High Energy Phys.* **05** (2003) 022 [hep-th/0210102].

[10] H. Verlinde, *Bits, matrices and 1/N*, hep-th/0206059;
J.-G. Zhou, *PP-wave string interactions from string bit model*, *Phys. Rev.* **D 67** (2003) 026010 [hep-th/0208232];
D. Vaman and H. Verlinde, *Bit strings from $N = 4$ gauge theory*, hep-th/0209215.

[11] M. Spradlin and A. Volovich, *Superstring interactions in a PP-wave background*, *Phys. Rev.* **D 66** (2002) 086004 [hep-th/0204146]; *Superstring interactions in a PP-wave background, II*, *J. High Energy Phys.* **01** (2003) 036 [hep-th/0206073].

[12] R. Roiban, M. Spradlin and A. Volovich, *On light-cone sft contact terms in a plane wave*, hep-th/0211220.

[13] J. Gomis, S. Moriyama and J.-w. Park, *Sym description of SFT hamiltonian in a PP-wave background*, hep-th/0210153.

[14] U. Gürsoy, *Vector operators in the BMN correspondence*, *J. High Energy Phys.* **07** (2003) 048 [hep-th/0208041].

[15] M.-X. Huang, *Three point functions of $N = 4$ super Yang-Mills from light cone string field theory in PP-wave*, *Phys. Lett.* **B 542** (2002) 255 [hep-th/0205311].

[16] N. Beisert, *BMN operators and superconformal symmetry*, *Nucl. Phys.* **B 659** (2003) 79 [hep-th/0211032].

[17] W. Skiba, *Correlators of short multi-trace operators in $N = 4$ supersymmetric Yang-Mills*, *Phys. Rev.* **D 60** (1999) 105038 [hep-th/9907088].

[18] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products*, fifth ed., Academic Press, New York 1993.

[19] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *BMN gauge theory as a quantum mechanical system*, *Phys. Lett.* **B 558** (2003) 229 [hep-th/0212269].

JHEP10(2003)027